Developing a stochastic linear elastic material model and composite laminate theory including failures, using the fiber bundle–based approach - Part II: Stochastic Composite Laminate Theory

Vas L. M., Tamás P.

Accepted for publication in Journal of Composite Materials
Published in 2024
DOI: 10.1177/00219983241302780

DEVELOPING A STOCHASTIC LINEAR ELASTIC MATERIAL MODEL AND COMPOSITE LAMINATE THEORY INCLUDING FAILURES, USING THE FIBER BUNDLE–BASED APPROACH

Part II: Stochastic Composite Laminate Theory

by László Mihály VAS[1] and Péter TAMÁS[2]

[1]Department of Polymer Engineering, Faculty of Mechanical Engineering, Budapest University of Technology and Economics, Műegyetem rkp. 3., H-1111 Budapest, Hungary
Phone: +(36) (1) 463 1529, Fax: +(36) (1) 463 1527
vas@pt.bme.hu[1a]
[2]Department of Mechatronics, Optics and Engineering Informatics, Faculty of Mechanical Engineering, Budapest University of Technology and Economics, Műegyetem rkp. 3., H-1111 Budapest, Hungary
Phone: +(36) (1) 463 1691, Fax: +(36) (1) 463 1689
tamas@mogi.bme.hu[2]

Notation

Abbreviations
CLT – Composite Laminate Theory
CCLT – Classical Composite Laminate Theory
DMR – Duplex Memory Reliability (function, tensor, matrix, …)
CP1, CP2 – Code of the Composite Plate specimens
FBC – Fiber Bundle Cells
FEM – Finite Element Method
GCS – Global Coordinate System
GMS – Global Memory Stiffness
LCS – Local Cartesian Coordinate System
MiF – Memory in-plane Force
MiM – Memory in-plane Moment
MSE – Mean Squared Error
MLE – Mean Linear Error
MRF – Memory Reliability Function
PDF – Probability Distribution Function
RF – Reliability Function
RMLE – Relative Mean Linear Error
RMSE – Relative Mean Squared Error
RVE – Representative Volume Element
SCLT – Stochastic Composite Laminate Theory
WF – Window Function
General signs and notations
a, *A* – cursive lowercase or uppercase letters symbolize real parameters, variables, or functions
$\underline{\mathrm{a}}$ – upright lowercase underlined letters symbolize vectors
A – upright regular uppercase letters symbolize matrix
B – general index for normal compressive (-*B*=*C*) or tensile (*B*=*T*) or shear (±*B*=±*S*) breaking strain
$\mathbb{D}$(X) – Standard Deviation of the stochastic variable X
$\mathbb{E}$(X) – Expected value of the stochastic variable X
$\widetilde{\mathrm{Y}}$, $\widetilde{Y}$– over-tilde~ designates the stochastic process character of matrix (Y) or composite function (Y) containing empirical duplex RFs (e.g. $r_{CT}(u)$ or $r^*_{CT}(u)$) as entries or internal variables, respectively

$\mathrm{Y}^*, \widetilde{\mathrm{Y}}^*$ – superscript asterisk* denotes the memory property of window functions (χ^*_{Zn}), empirical ($r^*_X(u)$) and expected reliability functions ($R^*_X(u)$), or stochastic vector or matrix (e.g. $\widetilde{\sigma}^*, \widetilde{\mathrm{C}}^*, \widetilde{\mathrm{R}}^*$). The expectation of the latter expressions has the memory property as well, therefore it is indicated by the asterisk only (e.g. $\sigma^*, \mathrm{C}^*, \mathrm{R}^*$).

Variables and parameters

$\underline{e}_i$ (i=1, 2, 3) – unit base vectors

$\underline{\mathrm{f}}^0, \underline{\mathrm{f}}^1$ – in-plane force and moment vectors, respectively

h – thickness of the laminate

r^*_{Li} (i=1,…,6) – empirical duplex MRF on the laminate level

$\underline{x}_{loc}$, $\underline{x}_{glob}$ – vector x in LCS and GCS, x∈{ε, σ}

A, B, W – general stiffness minor matrices of the laminate

C = [c_{ij}] – stiffness matrix of the lamina (c_{ij} are constant elements) in the LCS

$\widetilde{\mathrm{C}}^* = \mathrm{C}\widetilde{\mathrm{R}}^*$, $\mathrm{C}^* = \mathrm{CR}^*$ – empirical and expected memory stiffness matrices in the LCS

D^{*2} – variance matrix

E_i (*i*=1, 2) – tensile elastic moduli of the lamina

G_{12} – shear elastic modulus of the lamina

H_k thickness based geometrical hyper-matrix of the *k*th lamina

K – number of laminas

M – in-plane moment

N – in-plane force

Q – stiffness matrix of the lamina in the GCS

$\widetilde{\mathrm{Q}}^*$, Q^* – empirical and expected memory stiffness matrices of the lamina in the GCS

$\mathrm{Q}_{H,k}$ – stiffness hyper-matrix of the *k*th lamina

$\widetilde{\mathrm{R}}^* = [r^*_{ij}]$, $\mathrm{R}^* = [R^*_{ij}]$ – compacted empirical and expected DMR matrices of the lamina

$\widetilde{\mathrm{R}}^*_L$, R^*_L – empirical and expected DMR matrices on the laminate level

T(.) – rotation transformation matrix

(*x,x,x*) – Global Cartesian Coordinate System

(x_1, x_2, x_3) – Local Cartesian Coordinate System

γ – shear strain

$\chi^*_{-B,Bn}$ – duplex memory WF of the *n*th fiber related to –*B, B*∈{*CT; -S, S*}, respectively

Γ – general stiffness hyper-matrix of the laminate

$\widetilde{\Gamma}^*$, Γ^* – empirical and expected general memory stiffness hyper-matrices of the laminate

ε – normal strain

-ε_{-Bn}, ε_{Bn} – normal (*-B=C, B=T*) and shear (*B=S*) breaking strain of the *n*th bundle fiber, respectively

$\underline{\varepsilon} = [\varepsilon_i]$ – strain vector

ε^0_x, ε^0_x, γ^0_{xy} – in-plane normal and shear deformations that are the elements of vector $\underline{\varepsilon}^0$

κ^0_x, κ^0_x, κ^0_{xy} – out-of-plane deflection and torsion deformations that are the elements of vector $\underline{\varepsilon}^1$

ν_{ij} – Poisson's coefficients of the lamina

σ – normal stress

$\underline{\sigma} = [\sigma_i]$ – stress vector

θ – fiber orientation angle of the lamina in the laminate

τ – shear stress

Abstract

Using the fiber bundle cells (FBCs)-based stochastic material model developed in the first part of this paper, we derived a stochastic version of the classical composite laminate theory (SCLT) for controlled multiaxial deformation, where the strength parameters are stochastic variables. This model (SCLT) enables to describe both the deformation and the damage processes in each lamina up to the ultimate

failure. By discussing some examples of simpler tensile load, we demonstrate the applicability of the results.
Keywords: *stochastic material model, fiber bundle, composite laminate theory, memory reliability matrix, stochastic modeling*

1 INTRODUCTION

Designing fiber-reinforced composite structures has usually been based on the so-called Classical Composite Laminate Theory (CLT or CCLT)[1-3] or its extensions of different kinds, for example the first-, second-, or third-order shear deformable plate theories[3]. In general, the dimensioning of a construction or a machine part is performed using simulations realized in FEM (finite element method) software environments and to checking some failure criteria[4, 5] are applied as an extra operation. As another solution, the so called Cohesive Zone Method (CZM) can predict the crack or delamination and treat their effect on the behavior[6, 7]. The fiber bundles[8-10] and especially the fiber-bundle-cells (FBC)-based modeling method make it possible to describe the deformation and damage processes together when simple mechanical load is used[11-14].
In the first part of this paper[15], we introduced the notions of the memory window function of model fibers, memory reliability function (MRF) of the fiber bundles, and their duplex versions (DMR) to describe the irreversible effect of damages caused by alternating strain load. It was shown as well[15] that using the MRF provided a method more general than the CZM. Based on these notions and tools, we developed a stochastic linear material model that could take account the effect of damages up to the ultimate failure in the case of multiaxial normal and shear strain load as well[15].
Using this fiber bundle based material model, we elaborated a kind of stochastic version of the Classical Composite Laminate Theory (SCLT), and demonstrated its applicability with simple examples.

2 STOCHASTIC COMPOSITE LAMINATE THEORY

2.1 Classical Composite Laminate Theory (CCLT)

Fiber-reinforced composite plates or sheets are multilayer, that is, they are laminates, and the orientation of the fibers in the laminas or plies are different (Figure 1).
The mechanical material model of the layer-based laminate[2] is created corresponding to the structural levels in Figure 1.

- In the first step (Figure 1), the micromechanical stress–strain relationships related to the Representative Volume Element (RVE) of a single elementary layer (lamina or ply) are determined based on the assumed material properties of the fibers and matrix as elementary material components and following the methods of micromechanics.
- Based on the homogenization principle (H1), it is assumed that these relationships are true for every such volume element, the stress–strain relationship is understood for a homogeneous, orthotropic continuous layer in a local Cartesian coordinate system (LCS)(x_1,x_2,x_3) corresponding to the structural directions of the layer.
- The lamina equation is transformed into the global coordinate system (x,y,z) (GCS) of the laminate by rotation **T** with angle θ (Figure 2).
- The force and moment vs. deformation equation of the laminate can be obtained by integrating along the heights of the layers. Another homogenization step (H1), regarding the laminate a homogeneous but generally anisotropic continuum (Hom. Lam.) provides an invertible material equation for the deformation vs. force/moment relationship.

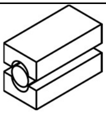
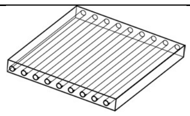
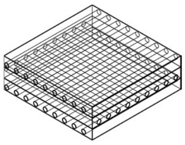
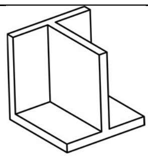
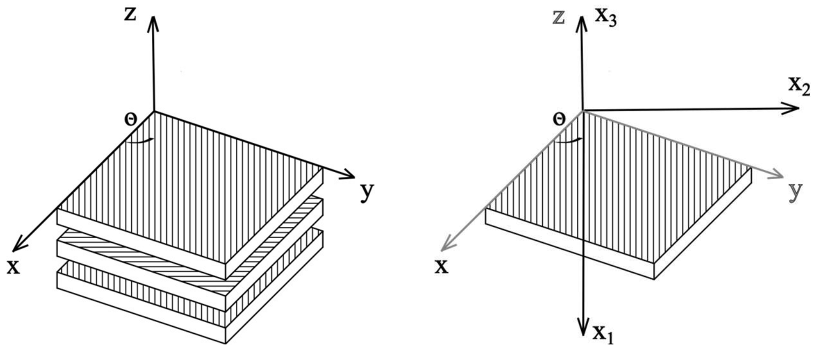
 Matrix and fiber	**RVE for lamina**	$\begin{Bmatrix} \boldsymbol{Fiber, Matrix} \\ \boldsymbol{RVE} \end{Bmatrix}$
↓ Micromechanics ↓	**Homogenization (H1)**	↓ **H1**
 Lamina (ply)	**Lamina/Ply/Layer** in local CS	$\begin{Bmatrix} \boldsymbol{Layer} \\ LCS \end{Bmatrix}$
↓ Macromechanics ↓	**Rotation (T)**	↓ **T**(θ)
 Laminate	**Lamina/Ply/Layer** in global CS	$\begin{Bmatrix} \boldsymbol{Layer} \\ GCS \end{Bmatrix}$
	Summation and integration	↓ $\sum \int$
	Laminate in global CS	$\begin{Bmatrix} \boldsymbol{Laminate} \\ GCS \end{Bmatrix}$
↓ Structural analysis ↓	**Homogenization (H2)**	↓ **H2**
 Structure	**Structure** made of laminate	$\begin{Bmatrix} \boldsymbol{Hom. Lam.} \\ GCS \end{Bmatrix}$

Figure 1 Structural level of the composite laminate and the flow chart of the process of model creation



(a) (b)

Figure 2 The x, y, z global (laminate) (a) and the x1, x2, x3 local (lamina/ply) (b) coordinate systems, and ply angle θ

The fundamental assumptions of the first order composite laminate theory are summarized in Chapter 3 of Kollar and Springer's book[2].

2.2 Stochastic Composite Laminate Theory (SCLT)

2.2.1 Material equation of a single lamina including damage

Local coordinate system

The *material equation including damage* in the local coordinate system of the lamina is a relationship between the local strain vector as stimulus and the local stress vector[2] (see Ref. 2, p.41), which contains the effects of the damage:

$$\underline{\tilde{\sigma}}^*_{loc} = \begin{bmatrix} \tilde{\sigma}_1^* \\ \tilde{\sigma}_2^* \\ \tilde{\tau}_{12}^* \end{bmatrix} = \begin{bmatrix} \frac{E_1}{H} & \frac{\nu_{12}E_2}{H} & 0 \\ \frac{\nu_{12}E_2}{H} & \frac{E_2}{H} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \begin{bmatrix} r_1^*(\varepsilon_1) & 0 & 0 \\ 0 & r_2^*(\varepsilon_2) & 0 \\ 0 & 0 & r_{12}^*(\gamma_{12}) \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} =$$

$$= \mathrm{C}\tilde{\mathrm{R}}^*(\underline{\varepsilon}_{loc})\underline{\varepsilon}_{loc} =: \tilde{\mathrm{C}}^*(\underline{\varepsilon}_{loc})\underline{\varepsilon}_{loc}, \qquad H = 1 - \frac{E_2}{E_1}\nu_{12}^2 = 1 - \nu_{12}\nu_{21} \quad (1)$$

where the elements of the ***empirical duplex memory reliability (DMR) matrix*** $\tilde{\mathrm{R}}^*$ are not constant but stochastic functions of the strain load elements:

$$\tilde{\mathrm{R}}^*(\underline{\varepsilon}_{loc}) = \begin{bmatrix} r_1^*(\varepsilon_1) & 0 & 0 \\ 0 & r_2^*(\varepsilon_2) & 0 \\ 0 & 0 & r_{12}^*(\gamma_{12}) \end{bmatrix}, \quad \tilde{\mathrm{C}}^*(\underline{\varepsilon}_{loc}) = \mathrm{C}\tilde{\mathrm{R}}^*(\underline{\varepsilon}_{loc}) \quad (2)$$

The expectation of the material Eq. (1), in which the elements of the ***expected DMR matrix*** R^* are also functions, is given by:

$$\underline{\sigma}^*_{loc} = \begin{bmatrix} \sigma_1^* \\ \sigma_2^* \\ \tau_{12}^* \end{bmatrix} = \begin{bmatrix} \frac{E_1}{H} & \frac{\nu_{12}E_2}{H} & 0 \\ \frac{\nu_{12}E_2}{H} & \frac{E_2}{H} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \begin{bmatrix} R_1^*(\varepsilon_1) & 0 & 0 \\ 0 & R_2^*(\varepsilon_2) & 0 \\ 0 & 0 & R_{12}^*(\gamma_{12}) \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} =$$

$$= \mathrm{C}\mathrm{R}^*(\underline{\varepsilon}_{loc})\underline{\varepsilon}_{loc} = \mathrm{C}^*(\underline{\varepsilon}_{loc})\underline{\boldsymbol{\varepsilon}}_{loc} \quad (3)$$

where

$$\mathrm{R}^*(\underline{\varepsilon}_{loc}) = \begin{bmatrix} R_1^*(\varepsilon_1) & 0 & 0 \\ 0 & R_2^*(\varepsilon_2) & 0 \\ 0 & 0 & R_{12}^*(\gamma_{12}) \end{bmatrix}, \quad \mathrm{C}^*(\underline{\varepsilon}_{loc}) = \mathrm{C}\mathrm{R}^*(\underline{\varepsilon}_{loc}) \quad (4)$$

Global coordinate system

In the global coordinate system, the stress–strain relationship of a single lamina including damage can be obtained by the rotation transformations $\mathrm{T}_\varepsilon(\theta)$ and $\mathrm{T}_\sigma(\theta)$ (see Chapter 3 in Ref. 2, p.69) where $s=\sin\theta$, $c=\cos\theta$). The fiber orientation angles, θ, in the laminas are stochastic variables, which are independent of each other.

The elements of the matrices $\tilde{\mathrm{R}}^*$ and R^* depend on the fiber deformations. Therefore, we have to take into account that these internal variables are a function of the controlled global variable when transforming into the global coordinate system. Correspondingly, the local variables as the transformations of the controlled global strain or stress are as follows[2] (see Ref. 2, p.68-69):

$$\underline{\varepsilon}_{loc} = \mathrm{T}_\varepsilon(\theta)\underline{\varepsilon}_{glob}, \quad \varepsilon_{loc,i} = \underline{e}_i^T \mathrm{T}_\varepsilon(\theta)\underline{\varepsilon}_{glob} \quad (5)$$

$$\underline{\sigma}_{loc} = \mathrm{T}_\sigma(\theta)\underline{\sigma}_{glob}, \quad \sigma_{loc,i} = \underline{e}_i^T \mathrm{T}_\sigma(\theta)\underline{\sigma}_{glob} \quad (6)$$

where $\underline{e}_i$ (i=1, 2, 3) is the i^{th} orthonormal unit base vector and $\varepsilon_{loc,i}$ and $\sigma_{loc,i}$.are the i^{th} component of vectors $\underline{\varepsilon}_{loc}$ and $\underline{\sigma}_{loc}$.

There are two ways of transforming the local material equations into the global system: we can directly transform the equations containing the stochastic processes or apply some intermediate expectation as well. In the latter case, we narrow the description of the local phenomena by statistical homogenization. As a result, it makes a sort of phenomenological description on the given level. With the use of Eqs. (2) and (6), the transformation of the material Eq. (3) with ***stochastic damage*** is:

$$\underline{\tilde{\sigma}}^*_{glob} = \mathrm{T}_\sigma^{-1}(\theta)\mathrm{C}\tilde{\mathrm{R}}^*(\mathrm{T}_\varepsilon(\theta)\underline{\varepsilon}_{glob})\mathrm{T}_\varepsilon(\theta)\underline{\varepsilon}_{glob} = \mathrm{T}_\sigma^{-1}\tilde{\mathrm{C}}^*(\mathrm{T}_\varepsilon(\theta)\underline{\varepsilon}_{glob})\mathrm{T}_\varepsilon\underline{\varepsilon}_{glob} = \tilde{\mathrm{Q}}^*(\underline{\varepsilon}_{glob})\underline{\varepsilon}_{glob} \quad (7)$$

where $\widetilde{\mathrm{Q}}^*$ is the empirical global memory stiffness matrix of the lamina:

$$\widetilde{\mathrm{Q}}^* = \widetilde{\mathrm{Q}}^*(\underline{\varepsilon}_{glob}) = \mathrm{T}_\sigma^{-1}(\theta)\mathrm{C}\widetilde{\mathrm{R}}^*(\mathrm{T}_\varepsilon(\theta)\underline{\varepsilon}_{glob})\mathrm{T}_\varepsilon(\theta) = \mathrm{T}_\sigma^{-1}(\theta)\tilde{\mathrm{C}}^*(\mathrm{T}_\varepsilon(\theta)\underline{\varepsilon}_{glob})\mathrm{T}_\varepsilon(\theta) \quad (8)$$

and $\tilde{\mathrm{C}}^* = \mathrm{C}\widetilde{\mathrm{R}}^*$. The ***expected*** material equation in the global coordinate system is given by:

$$\underline{\sigma}^*_{glob} = \begin{bmatrix} \sigma_x^* \\ \sigma_y^* \\ \tau_{xy}^* \end{bmatrix} = \mathbb{E}\left[\mathrm{T}_\sigma^{-1}(\theta)\mathrm{C}\widetilde{\mathrm{R}}^*(\mathrm{T}_\varepsilon(\theta)\underline{\varepsilon}_{glob})\mathrm{T}_\varepsilon(\theta)\right]\underline{\varepsilon}_{glob} = \mathbb{E}\left[\mathrm{T}_\sigma^{-1}\tilde{\mathrm{C}}^*\mathrm{T}_\varepsilon\right]\underline{\varepsilon}_{glob} = \mathrm{Q}^*(\underline{\varepsilon}_{glob})\underline{\varepsilon}_{glob} \quad (9)$$

where Q^* is the expected global memory stiffness matrix of the lamina:

$$\mathrm{Q}^* = \mathrm{Q}^*(\underline{\varepsilon}_{glob}) = \mathbb{E}\left[\widetilde{\mathrm{Q}}^*\right] = \mathbb{E}\left[\mathrm{T}_\sigma^{-1}\tilde{\mathrm{C}}^*(\mathrm{T}_\varepsilon\underline{\varepsilon}_{glob})\mathrm{T}_\varepsilon\right] = \mathbb{E}\left[\mathrm{T}_\sigma^{-1}(\theta)\mathrm{C}\widetilde{\mathrm{R}}^*(\mathrm{T}_\varepsilon(\theta)\underline{\varepsilon}_{glob})\mathrm{T}_\varepsilon(\theta)\right] \quad (10)$$

Regarding the elements of the matrix Q* characterizing the damage as well, all the above means that they depend on not only the local strain vector but also on every element of the global strain load vector.
Although the stochastic damage variables of $\widetilde{\mathrm{R}}^*$ are independent of the stochastic fiber orientation angle θ, but because of the transformation of variables, the matrix $\tilde{\mathrm{C}}^* = \mathrm{C}\widetilde{\mathrm{R}}^*$ also depends on θ, therefore the expectation given by Eq. (10) cannot be written in product form. However, applying the law of total expectation yields

$$\mathrm{Q}^* = \mathrm{Q}^*(\underline{\varepsilon}_{glob}) = \mathbb{E}\left[\widetilde{\mathrm{Q}}^*\right] = \mathbb{E}\left[\mathrm{T}_\sigma^{-1}\mathrm{C}\,\mathbb{E}\left(\widetilde{\mathrm{R}}^*(\mathrm{T}_\varepsilon\underline{\varepsilon}_{glob})\middle|\theta\right)\mathrm{T}_\varepsilon\right] \quad (11)$$

On the other hand, along the thickness of the lamina, the fiber orientation angle may be regarded as ***constant*** in lamina, for example, the nominal value, hence Eq. (11) may be rewritten:

$$\mathrm{Q}^* = \mathrm{Q}^*(\underline{\varepsilon}_{glob}) = \mathbb{E}\left[\widetilde{\mathrm{Q}}^*\right] = \mathrm{T}_\sigma^{-1}\mathrm{C}\,\mathbb{E}\left(\widetilde{\mathrm{R}}^*(\mathrm{T}_\varepsilon\underline{\varepsilon}_{glob})\right)\mathrm{T}_\varepsilon = \mathrm{T}_\sigma^{-1}\mathrm{C}\,\mathrm{R}^*(\mathrm{T}_\varepsilon\underline{\varepsilon}_{glob})\mathrm{T}_\varepsilon \quad (12)$$

Actually, Eq. (12) is a conditional expectation related to the given lamina.

2.2.2 In-plane forces and moments in plane stress state with damages

In the case of a controlled strain load, the response is the empirical stress distribution $\tilde{\sigma}^*_{glob}$ affected by random damage, but at the macro-level, the empirical ***memory in-plane forces*** (MiF), *N*, and ***moments*** (MiM), *M*, per unit length may also be considered (Figure 3)[2] (see in-plane forces and moments in Ref.2, p.67-68):

$$\begin{cases} \widetilde{N}_x^* = \int_{-h/2}^{h/2}\tilde{\sigma}_x^* dz, \quad \widetilde{N}_y^* = \int_{-h/2}^{h/2}\tilde{\sigma}_y^* dz, \quad \widetilde{N}_{xy}^* = \int_{-h/2}^{h/2}\tilde{\tau}_{xy}^* dz \\ \widetilde{M}_x^* = \int_{-h/2}^{h/2}z\tilde{\sigma}_x^* dz, \quad \widetilde{M}_y^* = \int_{-h/2}^{h/2}z\tilde{\sigma}_y^* dz, \quad \widetilde{M}_{xy}^* = \int_{-h/2}^{h/2}z\tilde{\tau}_{xy}^* dz \end{cases} \quad (13)$$

where, because of the plane stress state, the forces and moments perpendicular to the reference plane were neglected.

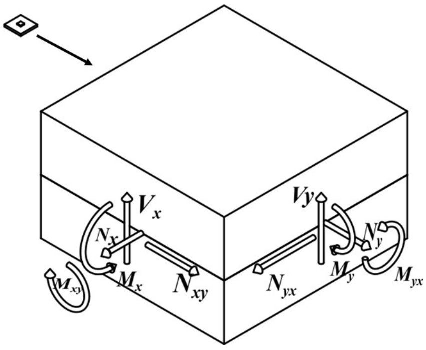


Figure 3 The in-plane forces (left) and the in-plane moments and the transverse shear forces (right) acting on the reference plane

Their expected values are the ***expected*** **MiF** and **MiM**:

$$\begin{cases} N_x^* = \int_{-h/2}^{h/2} \sigma_x^* dz, \quad N_y^* = \int_{-h/2}^{h/2} \sigma_y^* dz, \quad N_{xy}^* = \int_{-h/2}^{h/2} \tau_{xy}^* dz \\ M_x^* = \int_{-h/2}^{h/2} z\sigma_x^* dz, \quad M_y^* = \int_{-h/2}^{h/2} z\sigma_y^* dz, \quad M_{xy}^* = \int_{-h/2}^{h/2} z\tau_{xy}^* dz \end{cases} \tag{14}$$

where we used the short designations below like these

$$N_x^* = \mathbb{E}[\tilde{N}_x^*], \; M_x^* = \mathbb{E}[\tilde{M}_x^*] \tag{15}$$

$$\sigma_x^* = \mathbb{E}[\tilde{\sigma}_x^*], \; \tau_{xy}^* = \mathbb{E}[\tilde{\tau}_{xy}^*] \tag{16}$$

2.2.3 In-plane force/moment vs. strain/curvature relationship including damages

Actually, the fiber orientation angle is a stochastic process meaning that its parameters such as its expected value and variance may change along the thickness of the laminate (Figure 4).

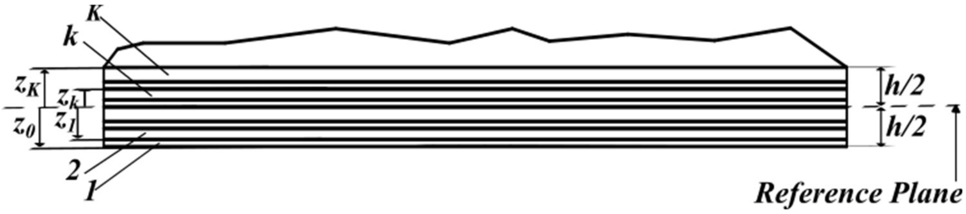


Figure 4 The general position of the reference plane and the thickness coordinates of the plies

Substituting Eq. (7) into Eq. (13) provides the relationship for the MiF ($\underline{\tilde{\mathrm{f}}}^{0*}$) and MiM ($\underline{\tilde{\mathrm{f}}}^{1*}$) vector variables vs. the *reference strain vector* ($\underline{\varepsilon}^0$) and the *curvature vector* of the reference plane ($\underline{\varepsilon}^1$)[2] (see Eqs. (3.16) and (3.17) in Ref.2, p.70):

$$\underline{\tilde{\mathrm{f}}}^{0*} = \begin{bmatrix} \tilde{N}_x^* \\ \tilde{N}_y^* \\ \tilde{N}_{xy}^* \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \underline{\tilde{\sigma}}_{glob}^* dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tilde{\mathrm{Q}}^*(\underline{\varepsilon}_{glob}) dz \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + \int_{-\frac{h}{2}}^{\frac{h}{2}} z\tilde{\mathrm{Q}}^*(\underline{\varepsilon}_{glob}) dz \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} =$$

$$= \tilde{\mathrm{A}}^* \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + \tilde{\mathrm{W}}^* \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \tilde{\mathrm{A}}^* \underline{\varepsilon}^0 + \tilde{\mathrm{W}}^* \underline{\varepsilon}^1 \tag{17}$$

$$\underline{\tilde{\mathrm{f}}}^{1*} = \begin{bmatrix} \tilde{M}_x^* \\ \tilde{M}_y^* \\ \tilde{M}_{xy}^* \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\underline{\tilde{\sigma}}_{glob}^* dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\tilde{\mathrm{Q}}^*(\underline{\varepsilon}_{glob}) dz \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2\tilde{\mathrm{Q}}^*(\underline{\varepsilon}_{glob}) dz \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} =$$

$$= \widetilde{\mathrm{W}}^* \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + \widetilde{\mathrm{B}}^* \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \widetilde{\mathrm{W}}^* \underline{\varepsilon}^0 + \widetilde{\mathrm{B}}^* \underline{\varepsilon}^1 \tag{18}$$

Taking into consideration that the parameters of the fiber orientation angle may be considered constant in the laminas, the new empirical memory stiffness matrices of the laminate that we introduced are as follows[2] (see Ref.2, p.70-71):

$$\widetilde{\mathrm{A}}^* = \int_{-h/2}^{h/2} \widetilde{\mathrm{Q}}^*(\underline{\varepsilon}_{glob})dz = \sum_{k=1}^{K} \int_{z_{k-1}}^{z_k} \widetilde{\mathrm{Q}}_k^*(\underline{\varepsilon}_{glob})dz = \sum_{k=1}^{K} \widetilde{\mathrm{Q}}_k^*(z_k - z_{k-1}) \tag{19}$$

$$\widetilde{\mathrm{W}}^* = \int_{-h/2}^{h/2} z\widetilde{\mathrm{Q}}^*(\underline{\varepsilon}_{glob})dz = \sum_{k=1}^{K} \int_{z_{k-1}}^{z_k} z\widetilde{\mathrm{Q}}_k^*(\underline{\varepsilon}_{glob})dz = \sum_{k=1}^{K} \widetilde{\mathrm{Q}}_k^* \frac{1}{2}(z_k^2 - z_{k-1}^2) \tag{20}$$

$$\widetilde{\mathrm{B}}^* = \int_{-h/2}^{h/2} z^2\widetilde{\mathrm{Q}}^*(\underline{\varepsilon}_{glob})dz = \sum_{k=1}^{K} \int_{z_{k-1}}^{z_k} z^2\widetilde{\mathrm{Q}}_k^*(\underline{\varepsilon}_{glob})dz = \sum_{k=1}^{K} \widetilde{\mathrm{Q}}_k^* \frac{1}{3}(z_k^3 - z_{k-1}^3) \tag{21}$$

where K is the number of the laminas and within the k^{th} lamina, the fiber orientation angle, θ_k, is constant, hence so is matrix $\widetilde{\mathrm{Q}}_k^*$, which is (from Eq. (8)):

$$\widetilde{\mathrm{Q}}_k^* = \mathrm{T}_{\sigma,k}^{-1}\mathrm{C}\widetilde{\mathrm{R}}_k^*\mathrm{T}_{\varepsilon,k} \tag{22}$$

Calculating the expectation of the empirical MiF and MiM vector given by Eqs. (17) and (18) and taking into consideration Eq. (9) yields:

$$\underline{\mathrm{f}}^{0*} = \mathbb{E}\left[\underline{\tilde{\mathrm{f}}}^{0*}\right] = \begin{bmatrix} N_x^* \\ N_y^* \\ N_{xy}^* \end{bmatrix} = \int_{-h/2}^{h/2} \mathbb{E}\left[\underline{\tilde{\sigma}}_{glob}^*\right]dz = \mathrm{A}^* \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + \mathrm{W}^* \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \mathrm{A}^*\underline{\varepsilon}^0 + \mathrm{W}^*\underline{\varepsilon}^1 \tag{23}$$

$$\underline{\mathrm{f}}^{1*} = \mathbb{E}\left[\underline{\tilde{\mathrm{f}}}^{1*}\right] = \begin{bmatrix} M_x^* \\ M_y^* \\ M_{xy}^* \end{bmatrix} = \int_{-h/2}^{h/2} z\mathbb{E}\left[\underline{\tilde{\sigma}}_{glob}^*\right]dz = \mathrm{W}^* \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + \mathrm{B}^* \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \mathrm{W}^*\underline{\varepsilon}^0 + \mathrm{B}^*\underline{\varepsilon}^1 \tag{24}$$

where

$$\mathrm{A}^* = \mathbb{E}\left[\widetilde{\mathrm{A}}^*\right] = \int_{-h/2}^{h/2} \mathrm{Q}^*(\underline{\varepsilon}_{glob})dz = \sum_{k=1}^{K} \int_{z_{k-1}}^{z_k} \mathrm{Q}_k^*(\underline{\varepsilon}_{glob})dz = \sum_{k=1}^{K} \mathrm{Q}_k^*(z_k - z_{k-1}) \tag{25}$$

$$\mathrm{W}^* = \mathbb{E}\left[\widetilde{\mathrm{W}}^*\right] = \int_{-h/2}^{h/2} z\mathrm{Q}^*(\underline{\varepsilon}_{glob})dz = \sum_{k=1}^{K} \int_{z_{k-1}}^{z_k} z\mathrm{Q}_k^*(\underline{\varepsilon}_{glob})dz = \sum_{k=1}^{K} \mathrm{Q}_k^* \frac{1}{2}(z_k^2 - z_{k-1}^2) \tag{26}$$

$$\mathrm{B}^* = \mathbb{E}\left[\widetilde{\mathrm{B}}^*\right] = \int_{-h/2}^{h/2} z^2\mathrm{Q}^*(\underline{\varepsilon}_{glob})dz = \sum_{k=1}^{K} \int_{z_{k-1}}^{z_k} z^2\mathrm{Q}_k^*(\underline{\varepsilon}_{glob})dz = \sum_{k=1}^{K} \mathrm{Q}_k^* \frac{1}{3}(z_k^3 - z_{k-1}^3) \tag{27}$$

$$\mathrm{Q}_k^* = \mathbb{E}\left[\widetilde{\mathrm{Q}}_k^*\right] = \mathrm{T}_{\sigma,k}^{-1}\mathrm{C}\mathbb{E}\left[\widetilde{\mathrm{R}}_k^*\right]\mathrm{T}_{\varepsilon,k} = \mathrm{T}_{\sigma,k}^{-1}\mathrm{C}\mathrm{R}_k^*\mathrm{T}_{\varepsilon,k} \tag{28}$$

Because of the transformation of variables, the matrix $\tilde{\mathrm{C}}^* = \mathrm{C}\widetilde{\mathrm{R}}^*$ depends on the deformation vector as well. Moreover, after calculating the expected values, this is also true for the expectation matrix Q^*.

2.2.4 Expected value equations of SCLT including damage at the lamina level

Formally, when the empirical MRFs are defined at the ***lamina level***, hence similar to the equation of the CLT[2] (see Ref.2, p.71), the MiF and MiM vector response given by Eqs. (17) and (18) to the controlled deformation load is as follows:

$$\underline{\varepsilon} \rightarrow \underline{\tilde{f}}^* = \begin{bmatrix} \underline{\tilde{f}}^{0*} \\ \underline{\tilde{f}}^{1*} \end{bmatrix} =: \begin{bmatrix} \tilde{N}_x^* \\ \tilde{N}_y^* \\ \tilde{N}_{xy}^* \\ \tilde{M}_x^* \\ \tilde{M}_y^* \\ \tilde{M}_{xy}^* \end{bmatrix} = \begin{bmatrix} \tilde{A}^* & \tilde{W}^* \\ \tilde{W}^* & \tilde{B}^* \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} \tilde{A}^* & \tilde{W}^* \\ \tilde{W}^* & \tilde{B}^* \end{bmatrix} \begin{bmatrix} \underline{\varepsilon}^0 \\ \underline{\varepsilon}^1 \end{bmatrix} =: \tilde{\Gamma}^* \underline{\varepsilon} \qquad (29)$$

and the expectation of Eq. (29) is

$$\underline{f}^* = \mathbb{E}(\underline{\tilde{f}}^*) = \begin{bmatrix} \underline{f}^{0*} \\ \underline{f}^{1*} \end{bmatrix} =: \begin{bmatrix} N_x^* \\ N_y^* \\ N_{xy}^* \\ M_x^* \\ M_y^* \\ M_{xy}^* \end{bmatrix} = \begin{bmatrix} A^* & W^* \\ W^* & B^* \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} A^* & W^* \\ W^* & B^* \end{bmatrix} \begin{bmatrix} \underline{\varepsilon}^0 \\ \underline{\varepsilon}^1 \end{bmatrix} =: \Gamma^* \underline{\varepsilon} \qquad (30)$$

$\tilde{\Gamma}^*$ is the ***empirical general memory stiffness*** (emp. GMS) hyper-matrix on the laminate level determined by the thicknesses defined by α_k, β_k, and γ_k values based on Eqs. (25)-(27):

$$\tilde{\Gamma}^* = \begin{bmatrix} \tilde{A}^* & \tilde{W}^* \\ \tilde{W}^* & \tilde{B}^* \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^K \tilde{Q}_k^* \alpha_k & \sum_{k=1}^K \tilde{Q}_k^* \gamma_k \\ \sum_{k=1}^K \tilde{Q}_k^* \gamma_k & \sum_{k=1}^K \tilde{Q}_k^* \beta_k \end{bmatrix} = \sum_{k=1}^K \begin{bmatrix} \tilde{Q}_k^* \alpha_k & \tilde{Q}_k^* \gamma_k \\ \tilde{Q}_k^* \gamma_k & \tilde{Q}_k^* \beta_k \end{bmatrix} = \sum_{k=1}^K \begin{bmatrix} \tilde{Q}_k^* & 0 \\ 0 & \tilde{Q}_k^* \end{bmatrix} \begin{bmatrix} \alpha_k I & \gamma_k I \\ \gamma_k I & \beta_k I \end{bmatrix} \qquad (31)$$

and Γ^* is its expectation, that is, the ***expected general memory stiffness*** (exp. GMS) hyper-matrix

$$\Gamma^* = \begin{bmatrix} A^* & W^* \\ W^* & B^* \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^K Q_k^* \alpha_k & \sum_{k=1}^K Q_k^* \gamma_k \\ \sum_{k=1}^K Q_k^* \gamma_k & \sum_{k=1}^K Q_k^* \beta_k \end{bmatrix} = \sum_{k=1}^K \begin{bmatrix} Q_k^* \alpha_k & Q_k^* \gamma_k \\ Q_k^* \gamma_k & Q_k^* \beta_k \end{bmatrix} = \sum_{k=1}^K \begin{bmatrix} Q_k^* & 0 \\ 0 & Q_k^* \end{bmatrix} \begin{bmatrix} \alpha_k I & \gamma_k I \\ \gamma_k I & \beta_k I \end{bmatrix} \qquad (32)$$

where H_k is a *thickness-based* ***geometrical hyper-matrix*** in Eqs. (19)-(21) and I is the 3x3 identity matrix:

$$H_k = \begin{bmatrix} \alpha_k I & \gamma_k I \\ \gamma_k I & \beta_k I \end{bmatrix} = \begin{bmatrix} (z_k - z_{k-1}) I & \frac{1}{2}(z_k^2 - z_{k-1}^2) I \\ \frac{1}{2}(z_k^2 - z_{k-1}^2) I & \frac{1}{3}(z_k^3 - z_{k-1}^3) I \end{bmatrix} \qquad (33)$$

With the use of Eq. (169) (33), the short form of Eqs. (167) (31) and (168) (32) are:

$$\tilde{\Gamma}^* = \sum_{k=1}^K \tilde{Q}_{H,k}^* H_k, \qquad \tilde{Q}_{H,k}^* = \begin{bmatrix} \tilde{Q}_k^* & 0 \\ 0 & \tilde{Q}_k^* \end{bmatrix} \qquad (34)$$

$$\Gamma^* = \sum_{k=1}^K Q_{H,k}^* H_k, \qquad Q_{H,k}^* = \mathbb{E}(\tilde{Q}_{H,k}^*) = \begin{bmatrix} Q_k^* & 0 \\ 0 & Q_k^* \end{bmatrix} \qquad (35)$$

where $\tilde{Q}_{H,k}^*$ and $Q_{H,k}^*$ are the hyper-matrix extension of matrices $\tilde{Q}_k^*$ and Q_k^*.
On the one hand, Eqs. (29) and (30) for the laminate correspond to Eqs. (109) and (112) in Part I[15] for the laminas. On the other hand, Eq. (30) corresponds to the deterministic material equation of the laminate according to the CLT[2]. In the latter case, all the MRFs equal 1 and Γ is the general stiffness matrix of the CLT:

$$\underline{\mathrm{f}} = \begin{bmatrix} \underline{\mathrm{f}}^0 \\ \underline{\mathrm{f}}^1 \end{bmatrix} =: \begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} \mathrm{A} & \mathrm{W} \\ \mathrm{W} & \mathrm{B} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \begin{bmatrix} \mathrm{A} & \mathrm{W} \\ \mathrm{W} & \mathrm{B} \end{bmatrix} \begin{bmatrix} \underline{\varepsilon}^0 \\ \underline{\varepsilon}^1 \end{bmatrix} =: \Gamma \underline{\varepsilon} \tag{36}$$

where

$$\Gamma = \begin{bmatrix} \mathrm{A} & \mathrm{W} \\ \mathrm{W} & \mathrm{B} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{K} \mathrm{Q}_k \alpha_k & \sum_{k=1}^{K} \mathrm{Q}_k \gamma_k \\ \sum_{k=1}^{K} \mathrm{Q}_k \gamma_k & \sum_{k=1}^{K} \mathrm{Q}_k \beta_k \end{bmatrix} = \sum_{k=1}^{K} \begin{bmatrix} \mathrm{Q}_k \alpha_k & \mathrm{Q}_k \gamma_k \\ \mathrm{Q}_k \gamma_k & \mathrm{Q}_k \beta_k \end{bmatrix} = \sum_{k=1}^{K} \begin{bmatrix} \mathrm{Q}_k & \mathrm{O} \\ \mathrm{O} & \mathrm{Q}_k \end{bmatrix} \begin{bmatrix} \alpha_k \mathrm{I} & \gamma_k \mathrm{I} \\ \gamma_k \mathrm{I} & \beta_k \mathrm{I} \end{bmatrix} \tag{37}$$

and

$$\mathrm{Q}_k = \mathrm{T}_{\sigma,k}^{-1} \mathrm{C} \mathrm{T}_{\varepsilon,k} \tag{38}$$

The short form of Eq. (37) is:

$$\Gamma = \sum_{k=1}^{K} \mathrm{Q}_{H,k} \mathrm{H}_k, \qquad \mathrm{Q}_{H,k} = \begin{bmatrix} \mathrm{Q}_k & \mathrm{O} \\ \mathrm{O} & \mathrm{Q}_k \end{bmatrix} \tag{39}$$

where $\mathrm{Q}_{H,k}$ is the hyper-matrix extension of matrix Q_k.

2.2.5 Phenomenological approximation including damage on the laminate level

Following the method of the CLT, we assume that the reinforcing fiber structure can be sufficiently characterized with the nominal fiber orientation in every lamina. In this case, the transformation matrices are constant. In addition, we suppose that the deterministic Eq. (36) of the CLT describes the behavior of the laminate when it is intact, where matrix Γ is constant and the empirical duplex MRFs r_{Li}^* (i=1,…,6) can be interpreted on the laminate level (L). In other words, the damage of the laminate generated by the strain load components can be represented by E-bundles defined on the laminate level. This means that to every component, ε_i, ($\varepsilon_i \in \{\varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0, \kappa_x, \kappa_y, \kappa_{xy}\}$), of the strain load vector, $\underline{\varepsilon}$, breaking strains are assigned on the laminate level, ε_{-Bi}^0 and ε_{+Bi}^0, as stochastic variables of known distribution functions with which the next empirical duplex MRF similar to Eqs. (75) and (76) in Part I[15] can be defined:

$$r_{Li}^*(\varepsilon_i) = \sum_{n=1}^{N} \chi_{Li,-B,+B}^*(\varepsilon_i; -\varepsilon_{-B,in}^0, \varepsilon_{+B,in}^0) \tag{40}$$

where χ_{Li}^* is the related duplex memory window function. Hence, taking into consideration that the indices -B and +B correspond to the indices C and T or –S and S and using Eqs. (101) and (102) in Part I[15], we can write the expected duplex MRF on the laminate level as:

$$R_{Li}^*(\varepsilon_i) = R_{Li,-B}^*(\varepsilon_i) R_{Li,+B}^*(\varepsilon_i) = R_{Li,-B}\left(\min_{0 \le t' \le t} \varepsilon_i(t')\right) R_{Li,+B}\left(\max_{0 \le t' \le t} \varepsilon_i(t')\right) \tag{41}$$

As we have seen above, the expected reliability functions can be expressed with the known distribution functions of the breaking strains. Accordingly, the next equation, which corresponds to Eq. (29) ($\underline{\tilde{\mathrm{f}}}_L^* = \tilde{\Gamma}^* \underline{\varepsilon}$) and related to the controlled strain load ($\underline{\varepsilon}$), represents a kind of macroscale homogenization as well:

$$\underline{\tilde{\mathrm{f}}}_L^* = \begin{bmatrix} \tilde{N}_x^* \\ \tilde{N}_y^* \\ \tilde{N}_{xy}^* \\ \tilde{M}_x^* \\ \tilde{M}_y^* \\ \tilde{M}_{xy}^* \end{bmatrix} = \Gamma \begin{bmatrix} r_{L1}^*(\varepsilon_x^0)\varepsilon_x^0 \\ r_{L2}^*(\varepsilon_y^0)\varepsilon_y^0 \\ r_{L3}^*(\gamma_{xy}^0)\gamma_{xy}^0 \\ r_{L4}^*(\kappa_x)\kappa_x \\ r_{L5}^*(\kappa_y)\kappa_y \\ r_{L6}^*(\kappa_{xy})\kappa_{xy} \end{bmatrix} = \Gamma \begin{bmatrix} r_{L1}^* & 0 & 0 & 0 & 0 & 0 \\ 0 & r_{L2}^* & 0 & 0 & 0 & 0 \\ 0 & 0 & r_{L3}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & r_{L4}^* & 0 & 0 \\ 0 & 0 & 0 & 0 & r_{L5}^* & 0 \\ 0 & 0 & 0 & 0 & 0 & r_{L6}^* \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \Gamma \tilde{\mathrm{R}}_L^*(\underline{\varepsilon})\underline{\varepsilon} = \tilde{\Gamma}_L^*(\underline{\varepsilon})\underline{\varepsilon} \tag{42}$$

where $\tilde{\mathrm{R}}_L^*$ is the empirical DMR matrix on the laminate level. The expectation of Eq. (42) is:

$$\underline{\mathrm{f}}_L^* = \mathbb{E}(\underline{\tilde{\mathrm{f}}}_L^*) = \begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \Gamma \begin{bmatrix} R_{L1}^*(\varepsilon_x^0)\varepsilon_x^0 \\ R_{L2}^*(\varepsilon_y^0)\varepsilon_y^0 \\ R_{L3}^*(\gamma_{xy}^0)\gamma_{xy}^0 \\ R_{L4}^*(\kappa_x)\kappa_x \\ R_{L5}^*(\kappa_y)\kappa_y \\ R_{L6}^*(\kappa_{xy})\kappa_{xy} \end{bmatrix} = \Gamma \begin{bmatrix} R_{L1}^* & 0 & 0 & 0 & 0 & 0 \\ 0 & R_{L2}^* & 0 & 0 & 0 & 0 \\ 0 & 0 & R_{L3}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{L4}^* & 0 & 0 \\ 0 & 0 & 0 & 0 & R_{L5}^* & 0 \\ 0 & 0 & 0 & 0 & 0 & R_{L6}^* \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \Gamma \mathrm{R}_L^*(\underline{\varepsilon})\underline{\varepsilon} = \Gamma_L^*(\underline{\varepsilon})\underline{\varepsilon} \quad (43)$$

Here, Γ is the deterministic general stiffness hypermatrix and

$$\tilde{\Gamma}_L^* = \Gamma \tilde{\mathrm{R}}_L^*(\underline{\varepsilon}) \quad (44)$$

$$\Gamma_L^* = \mathbb{E}(\tilde{\Gamma}^*) = \Gamma \mathrm{R}_L^*(\underline{\varepsilon}) \quad (45)$$

Similar to Eq. (113) in Part I[15] related to laminas, the determinant of matrix R_L^* can be applied as a global reliability number on the laminate level:

$$0 \le R_{L,Glob}^* = det\mathrm{R}_L^* = \prod_{i=1}^{6} R_{Li}^*(\varepsilon_i) \le 1 \quad (46)$$

where ε_i is the i^{th} component of strain vector $\underline{\varepsilon}$. The force response $\underline{\tilde{\mathrm{f}}}_L^*$ is an approximation of $\underline{\tilde{\mathrm{f}}}^*$. For the sake of comparison, let us consider the difference of the lamina level–based and the macro-scale homogenized Eqs. (29) and (42), respectively, and assume that Γ is invertible :

$$\underline{\tilde{\mathrm{f}}}_L^* - \underline{\tilde{\mathrm{f}}}^* = (\tilde{\Gamma}_L^* - \tilde{\Gamma}^*)\underline{\varepsilon} = \Gamma(\tilde{\mathrm{R}}_L^* - \Gamma^{-1}\tilde{\Gamma}^*)\underline{\varepsilon} \quad 47)$$

where

$$\tilde{\mathrm{R}}_L^* - \Gamma^{-1}\tilde{\Gamma}^* = \tilde{\mathrm{R}}_L^*(\underline{\varepsilon}) - \sum_{k=1}^{K} \Gamma^{-1}\tilde{\mathrm{Q}}_{H,k}^*\mathrm{H}_k = \sum_{k=1}^{K}\left[\frac{1}{K}\tilde{\mathrm{R}}_L^*(\underline{\varepsilon}) - \Gamma^{-1}\tilde{\mathrm{Q}}_{H,k}^*\mathrm{H}_k\right] \quad (48)$$

and

$$\tilde{\mathrm{Q}}_{H,k}^* = \begin{bmatrix} \tilde{\mathrm{Q}}_k^* & \mathrm{O} \\ \mathrm{O} & \tilde{\mathrm{Q}}_k^* \end{bmatrix} = \begin{bmatrix} \mathrm{T}_{\sigma,k}^{-1} & \mathrm{O} \\ \mathrm{O} & \mathrm{T}_{\sigma,k}^{-1} \end{bmatrix} \begin{bmatrix} \mathrm{C} & \mathrm{O} \\ \mathrm{O} & \mathrm{C} \end{bmatrix} \begin{bmatrix} \tilde{\mathrm{R}}_k^* & \mathrm{O} \\ \mathrm{O} & \tilde{\mathrm{R}}_k^* \end{bmatrix} \begin{bmatrix} \mathrm{T}_{\varepsilon,k} & \mathrm{O} \\ \mathrm{O} & \mathrm{T}_{\varepsilon,k} \end{bmatrix} = \mathrm{T}_{\sigma,H,k}^{-1}\mathrm{C}_H\tilde{\mathrm{R}}_{H,k}^*\mathrm{T}_{\varepsilon,H,k} \quad (49)$$

where the diagonal hyper-matrix factors are defined by Eq. (49). According to Eqs. (48) and (49), one k^{th} part of $\tilde{\mathrm{R}}_L^*$ corresponds to $\tilde{\mathrm{Q}}_{H,k}^*$, including $\tilde{\mathrm{R}}_{H,k}^*$ on the lamina level. This stands for the expectations as well, meaning that the elements of the DMR matrix R_L^* defined on the laminate level may be constructed as the linear combinations of the elements of the memory reliability matrices R_k^* (k=1,…,K) defined on lamina levels. According to Eq. (3), these elements are R_1^*, R_2^*, and R_{12}^*.

2.3 Variance equations of the SCLT including damage

Knowing the variance matrix, we can determine the multivariate probability distribution function of the force response vector and the confidence range for the force–strain relationship.

2.3.1 E-bundle–based damage modeling on the lamina level

The variance matrix of the force response vector $\underline{\tilde{\mathrm{f}}}^*$ can be calculated as that of the stress vector with Eq. (118) in Part I[15]:

$$\mathrm{D}_{\mathrm{f}}^{*2} = \mathbb{D}^2(\underline{\tilde{\mathrm{f}}}^*) = \mathbb{E}\left[(\underline{\tilde{\mathrm{f}}}^* - \underline{\mathrm{f}}^*)(\underline{\tilde{\mathrm{f}}}^* - \underline{\mathrm{f}}^*)^T\right] = \mathbb{E}[\underline{\tilde{\mathrm{f}}}^*\underline{\tilde{\mathrm{f}}}^{*T}] - \underline{\mathrm{f}}^*\underline{\mathrm{f}}^{*T} \quad (50)$$

or detailing that by using Eq. (29):

$$D_{\sigma}^{*2} = \mathbb{E}\left[\left(\tilde{\Gamma}^{*} - \Gamma^{*}\right)\underline{\varepsilon}\,\underline{\varepsilon}^{T}\left(\tilde{\Gamma}^{*} - \Gamma^{*}\right)^{T}\right] = \mathbb{E}\left[\tilde{\Gamma}^{*}\underline{\varepsilon}\,\underline{\varepsilon}^{T}\tilde{\Gamma}^{*T}\right] - \Gamma^{*}\underline{\varepsilon}\,\underline{\varepsilon}^{T}\Gamma^{*T} \tag{51}$$

Taking into consideration Eqs. (22), (28), (34), (35), and (49) leads to:

$$\tilde{\Gamma}^{*} - \Gamma^{*} = \sum_{k=1}^{K}\left(\tilde{Q}_{H,k}^{*} - Q_{H,k}^{*}\right)H_{k} = \sum_{k=1}^{K}\begin{bmatrix}\tilde{Q}_{k}^{*} - Q_{k}^{*} & 0\\ 0 & \tilde{Q}_{k}^{*} - Q_{k}^{*}\end{bmatrix}H_{k} =$$

$$= \sum_{k=1}^{K} T_{\sigma,H,k}^{-1}C_{H}\left(\tilde{R}_{H,k}^{*} - R_{H,k}^{*}\right)T_{\varepsilon,H,k}H_{k} = \sum_{k=1}^{K} T_{\sigma,H,k}^{-1}C_{H}\begin{bmatrix}\tilde{R}_{k}^{*} - R_{k}^{*} & 0\\ 0 & \tilde{R}_{k}^{*} - R_{k}^{*}\end{bmatrix}T_{\varepsilon,H,k}H_{k} \tag{52}$$

which can be calculated and so can the variance matrix.

2.3.2 E-bundle–based damage modeling on the laminate level

Similarly to the calculation at the lamina level, the variance matrix of $\tilde{\underline{f}}_{L}^{*}$ is:

$$D_{f_L}^{*2} = \mathbb{D}^{2}\left(\tilde{\underline{f}}_{L}^{*}\right) = \mathbb{E}\left[\left(\tilde{\underline{f}}_{L}^{*} - \underline{f}_{L}^{*}\right)\left(\tilde{\underline{f}}^{*} - \underline{f}_{L}^{*}\right)^{T}\right] = \mathbb{E}\left[\tilde{\underline{f}}_{L}^{*}\tilde{\underline{f}}_{L}^{*T}\right] - \underline{f}_{L}^{*}\underline{f}_{L}^{*T} \tag{53}$$

or detailing that by using Eqs. (42)-(44):

$$D_{\sigma}^{*2} = \mathbb{E}\left[\left(\tilde{\Gamma}_{L}^{*} - \Gamma^{*}\right)\underline{\varepsilon}\,\underline{\varepsilon}^{T}\left(\tilde{\Gamma}^{*} - \Gamma^{*}\right)^{T}\right] = \Gamma\mathbb{E}\left[\left(\tilde{R}_{L}^{*} - R_{L}^{*}\right)\underline{\varepsilon}\,\underline{\varepsilon}^{T}\left(\tilde{R}_{L}^{*} - R_{L}^{*}\right)^{T}\right]\Gamma^{T} =$$

$$= \Gamma\left[\mathbb{E}\left[\tilde{R}_{L}^{*}\underline{\varepsilon}\,\underline{\varepsilon}^{T}\tilde{R}_{L}^{*T}\right] - R_{L}^{*}\underline{\varepsilon}\,\underline{\varepsilon}^{T}R_{L}^{*T}\right]\Gamma^{T} \tag{54}$$

Eq. (54) is much simpler than those given by Eqs. (51) and (52).

3 APPLICATIONS

3.1 Structure and data of a two-layer composite plate

Two model composite specimens CP1 and CP2 were used for demonstrating the applicability of the theory elaborated. The specimens CP1 and CP2 could be regarded as those cut out of a two-layer (*K*=2) laminate [0°/90°] in different directions given by the angles θ_1=0° and θ_1=45°, respectively (Figure 5). The laminate is a simple model of a common biaxial woven fabric–reinforced composite plate. It was assumed that the specimens were subjected to constant rate tensile strain loads in directions related to the laminate at room temperature: in the fiber direction (CP1: θ_1=0°) and diagonal direction (CP2: θ_1=45°) (Figure 5: θ_1 and θ_2 determine the axes *x* and *y*, respectively). The thickness of the laminate was *h*=1 mm while the structure and dimensions ($h_k=h/K$=0.5 mm; *k*=1, 2) of the layers were the same (Table 1).

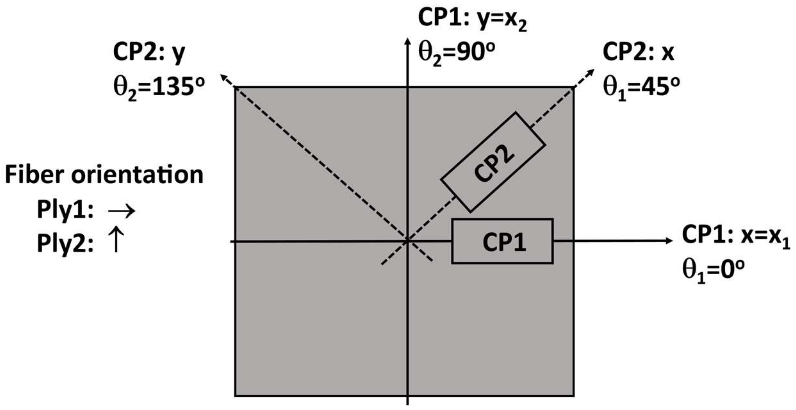


Figure 5 Specimens CP1 and CP2 cut out of the laminate and their loading directions (*x* and θ_1)

Laminate		**Parameters**			
Code	**Reinforcement**	**K**	$\mathbf{h_k}$	$\boldsymbol{\theta_1}$	$\boldsymbol{\theta_2}$
CP1	Biaxial	2	h/2	0°	90°
CP2	Biaxial	2	h/2	45°	135°

Table 1 Structural parameters of the composite plate

The lamina was considered a 2D orthotropic plate. Its stiffness matrix described by Eq. (2.139)[2] in Ref.2 (see p.41) contains the proper elastic engineering constants. For numerical calculations, the elastic constants of a graphite/epoxy unidirectional ply were taken from Kollar and Springer's book[2] (see the material named T300/934-tape in Table C.3, page 466) (Table 2) where ν_{21} was calculated from the following equation of symmetry:

$$\nu_{12}E_2 = \nu_{21}E_1 \quad \Rightarrow \quad \nu_{21} = \nu_{12}\frac{E_2}{E_1} \tag{55}$$

$\boldsymbol{E_1}$ [GPa]	$\boldsymbol{E_2}$ [GPa]	$\boldsymbol{G_{12}}$ [GPa]	$\boldsymbol{\nu_{12}}$ [-]	$\boldsymbol{\nu_{21}}$ [-]
148	9.65	4.55	0.3	0.0196

Table 2 Elastic constants of the composite plate

Thus, the numerical form of the stiffness matrix is (the unit of constants c_{ij} is GPa=10^3MPa)

$$\mathrm{C} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{bmatrix} = \begin{bmatrix} \frac{E_1}{H} & \frac{\nu_{12}E_2}{H} & 0 \\ \frac{\nu_{12}E_2}{H} & \frac{E_2}{H} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} = \begin{bmatrix} 148.88 & 2.91 & 0 \\ 2.91 & 9.71 & 0 \\ 0 & 0 & 4.55 \end{bmatrix}, \quad H = 0.9941 \tag{56}$$

The breaking strains are assumed to be stochastic variables of normal distribution. The mean breaking strain data (ε_B) were computed as the ratio of the strength[2] (see $S_{ij}^{\pm}$ in Ref.2, T300/934 tape in Table C.4, p.466) and the modulus (Table 2) values (ε_B=S/E). Standard deviation (SD) was calculated with assumed values of the coefficient of variation (CV) (Table 3).

Parameters	**Breaking strain**					
	ε_{C1} [%]	ε_{T1} [%]	ε_{C2} [%]	ε_{T2} [%]	ε_{-S12} [%]	ε_{+S12} [%]
Mean [%]	-0.82	0.89	-1.74	0.45	-1.05	1.05
CV [-]	0.05	0.05	0.10	0.10	0.15	0.15
SD [%]	0.041	0.044	0.174	0.045	0.158	0.158

Table 3 Statistical parameters of breaking strains

The laminate is built up of two identical laminas or plies rotated by 90° to each other. The matrix H_k of the k^{th} ply (k=1, 2) defined by Eq. (33) contained thickness-based geometrical properties only:

$$\mathrm{H}_k = \begin{bmatrix} \alpha_k & \gamma_k \\ \gamma_k & \beta_k \end{bmatrix} = \begin{bmatrix} z_k - z_{k-1} & z_k^2 - z_{k-1}^2 \\ z_k^2 - z_{k-1}^2 & z_k^3 - z_{k-1}^3 \end{bmatrix} \tag{57}$$

Its elements related to the two plies of the laminate (k=1, 2) are shown in Table 4:

H_k	α_k	γ_k	β_k
H_1	$\frac{h}{2} = \frac{1}{2}$	$-\frac{h^2}{8} = -\frac{1}{2}\left[\frac{h}{2}\right]^2 = -\frac{1}{8}$	$\frac{h^3}{24} = \frac{1}{3}\left[\frac{h}{2}\right]^3 = \frac{1}{24}$
H_2		$\frac{h^2}{8} = \frac{1}{2}\left[\frac{h}{2}\right]^2 = \frac{1}{8}$	

Table 4 Elements of matrix H_k

3.2 Relationships for the uniaxial tensile test

Applying uniaxial tensile strain load in the x direction, ε_x^0, as input on the laminate level makes the curvature part, $\underline{\varepsilon}^1$, of the strain vector, $\underline{\varepsilon}$, given by Eqs. (23) and (24) is zero ($\underline{\varepsilon}^1=\underline{0}$), therefore the output in-plane force/moment vector $\underline{\mathrm{f}}^*$ given by Eqs. (30) and (31) becomes

$$\underline{\mathrm{f}}^* = \begin{bmatrix} \underline{\mathrm{f}}^{0*} \\ \underline{\mathrm{f}}^{1*} \end{bmatrix} = \Gamma^* \underline{\varepsilon} = \begin{bmatrix} \mathrm{A}^* & \mathrm{W}^* \\ \mathrm{W}^* & \mathrm{B}^* \end{bmatrix} \begin{bmatrix} \underline{\varepsilon}^0 \\ \underline{\varepsilon}^1 \end{bmatrix} = \begin{bmatrix} \mathrm{A}^* \underline{\varepsilon}^0 \\ \mathrm{W}^* \underline{\varepsilon}^0 \end{bmatrix} = \begin{bmatrix} \left[\sum_{k=1}^2 \mathrm{Q}_k^* \alpha_k\right] \underline{\varepsilon}^0 \\ \left[\sum_{k=1}^2 \mathrm{Q}_k^* \gamma_k\right] \underline{\varepsilon}^0 \end{bmatrix} \tag{58}$$

where the two component vectors of $\underline{\mathrm{f}}^*$ are

$$\underline{\mathrm{f}}^{0*} = \begin{bmatrix} N_x^* \\ N_y^* \\ N_{xy}^* \end{bmatrix} = \left[\sum_{k=1}^2 \mathrm{Q}_k^* \alpha_k\right] \underline{\varepsilon}^0 = \left[\sum_{k=1}^2 \mathrm{Q}_k^* \alpha_k\right] \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_{xy}^0 \\ \gamma_{xy}^0 \end{bmatrix} = \left[\mathrm{Q}_1^* \alpha_1 + \mathrm{Q}_2^* \alpha_2\right] \begin{bmatrix} \varepsilon_x^0 \\ 0 \\ 0 \end{bmatrix} \tag{59}$$

$$\underline{\mathrm{f}}^{1*} = \begin{bmatrix} M_x^* \\ M_y^* \\ M_{xy}^* \end{bmatrix} = \left[\sum_{k=1}^2 \mathrm{Q}_k^* \gamma_k\right] \underline{\varepsilon}^0 = \left[\sum_{k=1}^2 \mathrm{Q}_k^* \gamma_k\right] \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_{xy}^0 \\ \gamma_{xy}^0 \end{bmatrix} = \left[\mathrm{Q}_1^* \gamma_1 + \mathrm{Q}_2^* \gamma_2\right] \begin{bmatrix} \varepsilon_x^0 \\ 0 \\ 0 \end{bmatrix} \tag{60}$$

The memory stiffness matrix of the k^{th} layer is given by Eqs. (148) (12) and (164) (28):

$$\mathrm{Q}_k^* = \begin{bmatrix} Q_{11,k}^* & Q_{12,k}^* & Q_{13,k}^* \\ Q_{21,k}^* & Q_{22,k}^* & Q_{23,k}^* \\ Q_{31,k}^* & Q_{32,k}^* & Q_{33,k}^* \end{bmatrix} = \mathrm{T}_\sigma^{-1}(\theta_k)\mathrm{C}\mathrm{R}_k^*\mathrm{T}_\varepsilon(\theta_k) = \mathrm{T}_\sigma^{-1}(\theta_k)\mathrm{C} \begin{bmatrix} R_{1,k}^* & 0 & 0 \\ 0 & R_{2,k}^* & 0 \\ 0 & 0 & R_{12,k}^* \end{bmatrix} \mathrm{T}_\varepsilon(\theta_k) =$$

$$= \mathrm{T}_\sigma^{-1}(\theta_k) \begin{bmatrix} c_{11} R_{1,k}^*(\varepsilon_{1,k}) & c_{12} R_{2,k}^*(\varepsilon_{2,k}) & 0 \\ c_{12} R_{1,k}^*(\varepsilon_{1,k}) & c_{22} R_{2,k}^*(\varepsilon_{2,k}) & 0 \\ 0 & 0 & c_{33} R_{12,k}^*(\gamma_{12,k}) \end{bmatrix} \mathrm{T}_\varepsilon(\theta_k) \tag{61}$$

where the i^{th} component of the local strain, $\underline{\varepsilon}_{loc,k}$, that is, the internal independent variable of the MRFs, $R_{j,k}^*$, can be obtained by multiplying with the unit base vector $\underline{e}_i$ (i=1, 2, 3):

$$\varepsilon_{i,k} = \underline{e}_i^T \underline{\varepsilon}_{loc,k} = \underline{e}_i^T \begin{bmatrix} \varepsilon_{1,k} \\ \varepsilon_{2,k} \\ \gamma_{12,k} \end{bmatrix} = \underline{e}_i^T \mathrm{T}_\varepsilon(\theta_k)\underline{\varepsilon}_{glob} = \underline{e}_i^T \mathrm{T}_\varepsilon(\theta_k) \begin{bmatrix} \varepsilon_x^0 \\ 0 \\ 0 \end{bmatrix} \tag{62}$$

Because of that, the MRFs $R_{j,k}^*$ depend on fiber orientation, θ_k, as well.
Based on Eqs. (60) and (61), we obtain that, in this simple uniaxial tensile test, we need to calculate only 3 components of the matrix Q_k^*:

$$\mathrm{Q}_k^* \begin{bmatrix} \varepsilon_x^0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Q_{11,k}^* \\ Q_{21,k}^* \\ Q_{31,k}^* \end{bmatrix} \varepsilon_x^0 \tag{63}$$

Making use of Eqs. (197) (61) and (3.15) in [2, p69], we can calculate these matrix elements:

$$Q_{11,k}^* = c^2(c^2c_{11} + s^2c_{12})R_{1,k}^* + s^2(c^2c_{12} + s^2c_{22})R_{2,k}^* + 4c^2s^2c_{33}R_{12,k}^* \tag{64}$$
$$Q_{21,k}^* = c^2(s^2c_{11} + c^2c_{12})R_{1,k}^* + s^2(s^2c_{12} + c^2c_{22})R_{2,k}^* - 4c^2s^2c_{33}R_{12,k}^* \tag{65}$$
$$Q_{31,k}^* = c^3s(c_{11} - c_{12})R_{1,k}^* + cs^3(c_{12} - c_{22})R_{2,k}^* - 2cs(c^2 - s^2)^2c_{33}R_{12,k}^* \tag{66}$$

According to Eq. (62), the monotone increasing strain load acting on the laminate level generates monotone increasing local strains in the laminas, therefore their maximum in time is equal to themselves. Consequently, based on Eqs. (89)-(94) in Part I[15], we can apply normal reliability functions instead of the memory types:

$$R_{1,k}^*(\varepsilon_{1,k}) = R_{C1,k}^*(\varepsilon_{1,k})R_{T1,k}^*(\varepsilon_{1,k}) = R_{C1,k}(\varepsilon_{1,k})R_{T1,k}(\varepsilon_{1,k}) = R_{CT1,k}(\varepsilon_{1,k})$$
$$= P_{-\varepsilon_{C1},k}(\varepsilon_{1,k})\left(1 - P_{\varepsilon_{T1},k}(\varepsilon_{1,k})\right) \tag{67}$$
$$R_{2,k}^*(\varepsilon_{2,k}) = R_{C2,k}^*(\varepsilon_{2,k})R_{T2,k}^*(\varepsilon_{2,k}) = R_{C2,k}(\varepsilon_{2,k})R_{T2,k}(\varepsilon_{2,k}) = R_{CT2,k}(\varepsilon_{2,k})$$
$$= P_{-\varepsilon_{C2},k}(\varepsilon_{2,k})\left(1 - P_{\varepsilon_{T2},k}(\varepsilon_{2,k})\right) \tag{68}$$
$$R_{12,k}^*\left(\gamma_{12,k}\right) = R_{-S,k}^*\left(\gamma_{12,k}\right)R_{S,k}^*\left(\gamma_{12,k}\right) = R_{-S12,k}\left(\gamma_{12,k}\right)R_{S12,k}\left(\gamma_{12,k}\right) = R_{-S,S,12,k}\left(\gamma_{12,k}\right)$$
$$= P_{-\varepsilon_{-S12},k}\left(\gamma_{12,k}\right)\left(1 - P_{\varepsilon_{S1}\ ,k}\left(\gamma_{12,k}\right)\right) \tag{69}$$

3.3 Modeling damage at the lamina level

3.3.1 Uniaxial tensile test of the laminate in the fiber direction

Local deformations of the two layers in specimen CP1 as the response to the global strain load are[2] (see Ref.2, p.51):

$$\underline{\varepsilon}_{loc,1} = \begin{bmatrix} \varepsilon_{1,1} \\ \varepsilon_{2,1} \\ \varepsilon_{12,1} \end{bmatrix} = \mathrm{T}_\varepsilon(0^o)\underline{\varepsilon}_{glob} = \mathrm{T}_\varepsilon(0^o)\underline{\varepsilon}^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \varepsilon_x^0 \\ 0 \\ 0 \end{bmatrix} = \varepsilon_x^0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{70}$$
$$\underline{\varepsilon}_{loc,2} = \begin{bmatrix} \varepsilon_{1,2} \\ \varepsilon_{2,2} \\ \varepsilon_{12,2} \end{bmatrix} = \mathrm{T}_\varepsilon(90^o)\underline{\varepsilon}_{glob} = \mathrm{T}_\varepsilon(0^o)\underline{\varepsilon}^0 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \varepsilon_x^0 \\ 0 \end{bmatrix} = \varepsilon_x^0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{71}$$

Using Eqs. (64)-(66), we can calculate the matrix elements needed:
$\mathrm{T}_\varepsilon(0^o)$: c=1, s=0;

$$Q_1^* \begin{bmatrix} \varepsilon_x^0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Q_{11,1}^* \\ Q_{21,1}^* \\ Q_{31,1}^* \end{bmatrix} \varepsilon_x^0 = \begin{bmatrix} c_{11} R_{1,1}^*(\varepsilon_{1,1}) \\ c_{12} R_{1,1}^*(\varepsilon_{1,1}) \\ 0 \end{bmatrix} \varepsilon_x^0 = \begin{bmatrix} c_{11} R_{1,1}^*(\varepsilon_x^0) \\ c_{12} R_{1,1}^*(\varepsilon_x^0) \\ 0 \end{bmatrix} \varepsilon_x^0 \tag{72}$$

$T_\varepsilon(90^o)$: c=0, s=1;

$$Q_2^* \begin{bmatrix} \varepsilon_x^0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Q_{11,2}^* \\ Q_{21,2}^* \\ Q_{31,2}^* \end{bmatrix} \varepsilon_x^0 = \begin{bmatrix} c_{22} R_{2,2}^*(\varepsilon_{2,2}) \\ c_{12} R_{2,2}^*(\varepsilon_{2,2}) \\ 0 \end{bmatrix} \varepsilon_x^0 = \begin{bmatrix} c_{22} R_{2,2}^*(\varepsilon_x^0) \\ c_{12} R_{2,2}^*(\varepsilon_x^0) \\ 0 \end{bmatrix} \varepsilon_x^0 \tag{73}$$

So, utilizing Table 4, the response force vector components given by Eqs. (59) and (60) are as follows:

$$\underline{f}^{0*} = \begin{bmatrix} N_x^* \\ N_y^* \\ N_{xy}^* \end{bmatrix} = [Q_1^* \alpha_1 + Q_2^* \alpha_2] \begin{bmatrix} \varepsilon_x^0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_{11} R_{1,1}^*(\varepsilon_x^0) + c_{22} R_{2,2}^*(\varepsilon_x^0) \\ c_{12}\left(R_{1,1}^*(\varepsilon_x^0) + R_{2,2}^*(\varepsilon_x^0)\right) \\ 0 \end{bmatrix} \frac{h}{2} \varepsilon_x^0 \tag{74}$$

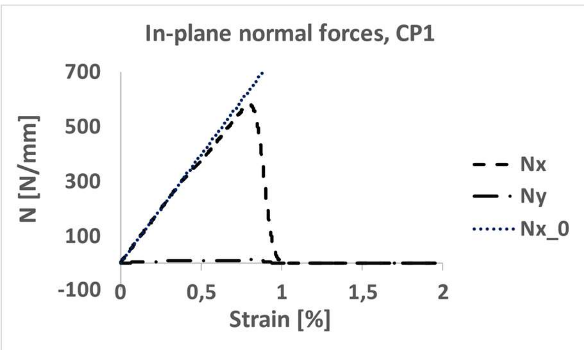
$$\underline{f}^{1*} = \begin{bmatrix} M_x^* \\ M_y^* \\ M_{xy}^* \end{bmatrix} = [Q_1^* \gamma_1 + Q_2^* \gamma_2] \begin{bmatrix} \varepsilon_x^0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_{22} R_{2,2}^*(\varepsilon_x^0) - c_{11} R_{1,1}^*(\varepsilon_x^0) \\ c_{12}\left(R_{2,2}^*(\varepsilon_x^0) - R_{1,1}^*(\varepsilon_x^0)\right) \\ 0 \end{bmatrix} \frac{h^2}{8} \varepsilon_x^0 \tag{75}$$

Accordingly, when the strain load is applied in x_1 direction of the first lamina, that is, the machine or warp direction of the woven fabric, the MiF N_{xy}^* and MiM M_{xy}^* are zero. Eqs. (74) and (75) describe both the deformation and the damage/failure processes in the laminate at an arbitrary strain load. However, when the strain load is small enough, the value of every reliability function is approximately 1. In this case, Eqs. (74) and (75) become much simpler (h=1 mm):

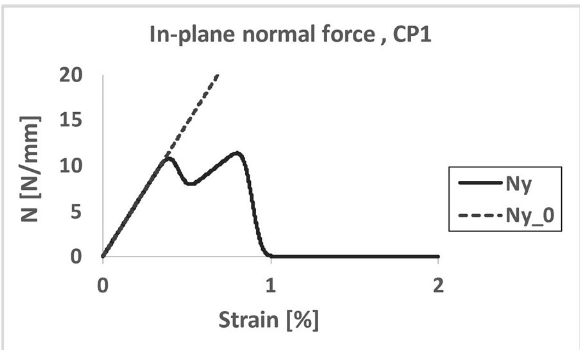
$$\underline{f}^{0*} = \begin{bmatrix} N_x^* \\ N_y^* \\ N_{xy}^* \end{bmatrix} \approx \begin{bmatrix} c_{11} + c_{22} \\ 2c_{12} \\ 0 \end{bmatrix} \frac{h}{2} \varepsilon_x^0 = \begin{bmatrix} 158.58 \\ 5.82 \\ 0 \end{bmatrix} \frac{h}{2} \varepsilon_x^0 = \begin{bmatrix} 79.29 \\ 2.91 \\ 0 \end{bmatrix} \varepsilon_x^0 \tag{76}$$

$$\underline{f}^{1*} = \begin{bmatrix} M_x^* \\ M_y^* \\ M_{xy}^* \end{bmatrix} \approx \begin{bmatrix} c_{22} - c_{11} \\ 0 \\ 0 \end{bmatrix} \frac{h^2}{8} \varepsilon_x^0 = \begin{bmatrix} -139.17 \\ 0 \\ 0 \end{bmatrix} \frac{h^2}{8} \varepsilon_x^0 = \begin{bmatrix} -17.40 \\ 0 \\ 0 \end{bmatrix} \varepsilon_x^0 \tag{77}$$

Obviously, when the lamina material is isotropic, hence $c_{11}=c_{22}$, no in-plane moment arises in the laminate. Figures 6 and 7 show the force and moment elements of $\underline{f}^{0*}$ and $\underline{f}^{1*}$ given by Eqs. (74) and (75), respectively, as functions of the strain load ε_x^0. The initial tangents (dashed lines) calculated with Eqs. (76) and (77) correspond to functioning without damage.



(a)

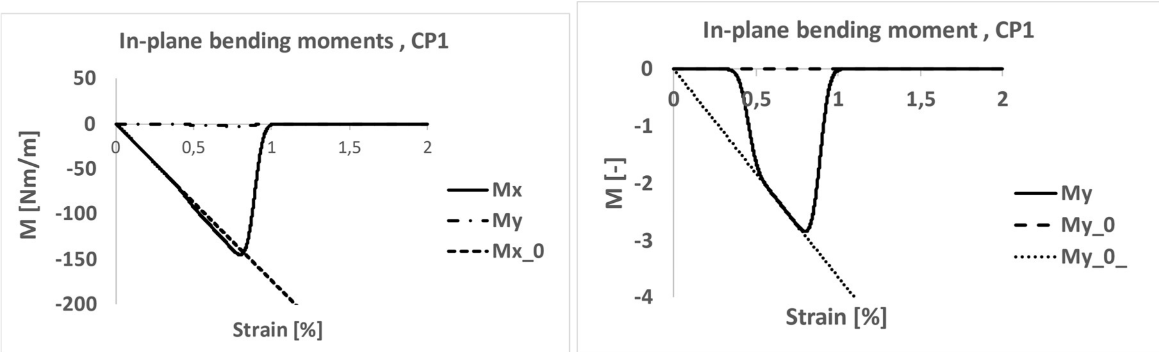


(b)

Figure 6 In-plane normal force functions N_x and N_y for specimen CP1 (a) and the enlarged N_y (b)

According to Eqs. (74) and (75), the normal forces and bending moments arising in the laminate depend on the state influenced by the damage, thus damage-induced fluctuations are possible (Figure 6.b). Otherwise, when each reliability is 1, there is no moment M_y as it is shown by Eqs. (76) and (77).

Because of the different elastic parameters of the laminas in directions 1 and 2, M_x moment may occur. However, the different strength properties in directions 1 and 2 may cause not only a nonzero expected value of moment M_x but a nonzero expectation of moment M_y as well (Figure 7.b). They are induced by the stochastic damage and failure processes.



(a) (b)

Figure 7 In-plane bending moment functions M_x and M_y for specimen CP1 (a) and the enlarged M_y (b)

The diagrams show the whole deformation and damage/failure process of the composite plate.

3.3.2 Uniaxial tensile test of the laminate in the diagonal direction

Local deformations of the two layers in specimen CP2 as the response to the global strain load are

$$\underline{\varepsilon}_{loc,1} = \begin{bmatrix} \varepsilon_{1,1} \\ \varepsilon_{2,1} \\ \varepsilon_{12,1} \end{bmatrix} = \mathrm{T}_\varepsilon(45^o)\underline{\varepsilon}_{glob} = \mathrm{T}_\varepsilon(45^o)\underline{\varepsilon}^0 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -2 & 2 & 0 \end{bmatrix}\begin{bmatrix} \varepsilon_x^0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_x^0 \\ -2\varepsilon_x^0 \end{bmatrix} = \frac{\varepsilon_x^0}{2}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad (78)$$

$$\underline{\varepsilon}_{loc,2} = \begin{bmatrix} \varepsilon_{1,2} \\ \varepsilon_{2,2} \\ \varepsilon_{12,2} \end{bmatrix} = \mathrm{T}_\varepsilon(135^o)\underline{\varepsilon}_{glob} = \mathrm{T}_\varepsilon(135^o)\underline{\varepsilon}^0 = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} \varepsilon_x^0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_x^0 \\ 2\varepsilon_x^0 \end{bmatrix} = \frac{\varepsilon_x^0}{2}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad (79)$$

Using Eqs. (64)-(66), we can calculate the matrix elements needed:
$\mathrm{T}_\varepsilon(45^o)$: $c = \frac{1}{\sqrt{2}},\ s = \frac{1}{\sqrt{2}}$;

$$Q_{11,1}^* = \frac{1}{4}\left[(c_{11}+c_{12})R_{1,1}^* + (c_{12}+c_{22})R_{2,1}^* + 4c_{33}R_{12,1}^*\right] \quad (80)$$
$$Q_{21,1}^* = \frac{1}{4}\left[(c_{11}+c_{12})R_{1,1}^* + (c_{12}+c_{22})R_{2,1}^* - 4c_{33}R_{12,1}^*\right] \quad (81)$$
$$Q_{31,1}^* = \frac{1}{4}(c_{11}-c_{12})\left[R_{1,1}^* + R_{2,1}^*\right] \quad (82)$$

$\mathrm{T}_\varepsilon(135^o)$: $c = -\frac{1}{\sqrt{2}},\ s = \frac{1}{\sqrt{2}}$;

$$Q_{11,2}^* = \frac{1}{4}\left[(c_{11}+c_{12})R_{1,2}^* + (c_{12}+c_{22})R_{2,2}^* + 4c_{33}R_{12,2}^*\right] \quad (83)$$
$$Q_{21,2}^* = \frac{1}{4}\left[(c_{11}+c_{12})R_{1,2}^* + (c_{12}+c_{22})R_{2,2}^* - 4c_{33}R_{12,2}^*\right] \quad (84)$$
$$Q_{31,2}^* = -\frac{1}{4}(c_{11}-c_{12})\left[R_{1,2}^* + R_{2,2}^*\right] \quad (85)$$

Finally, the vectors of the MiFs and MiMs are as follows:

$$\underline{f}^{0*} = \begin{bmatrix} N_x^* \\ N_y^* \\ N_{xy}^* \end{bmatrix} = \begin{bmatrix} (c_{11}+c_{12})(R_{1,1}^*+R_{1,2}^*)+(c_{12}+c_{22})(R_{2,1}^*+R_{2,2}^*)+4c_{33}(R_{12,1}^*+R_{12,2}^*) \\ (c_{11}+c_{12})(R_{1,1}^*+R_{1,2}^*)+(c_{12}+c_{22})(R_{2,1}^*+R_{2,2}^*)-4c_{33}(R_{12,1}^*+R_{12,2}^*) \\ (c_{11}-c_{12})[(R_{1,1}^*-R_{1,2}^*)+(R_{2,1}^*-R_{2,2}^*)] \end{bmatrix} \frac{h}{8}\varepsilon_x^0 \quad (86)$$

$$\underline{f}^{1*} = \begin{bmatrix} M_x^* \\ M_y^* \\ M_{xy}^* \end{bmatrix} = \begin{bmatrix} (c_{11}+c_{12})(R_{1,2}^*-R_{1,1}^*)+(c_{12}+c_{22})(R_{2,2}^*-R_{2,1}^*)+4c_{33}(R_{12,2}^*-R_{12,1}^*) \\ (c_{11}+c_{12})(R_{1,2}^*-R_{1,1}^*)+(c_{12}+c_{22})(R_{2,2}^*-R_{2,1}^*)-4c_{33}(R_{12,2}^*-R_{12,1}^*) \\ -(c_{11}-c_{12})[(R_{1,1}^*+R_{1,2}^*)+(R_{2,2}^*+R_{2,1}^*)] \end{bmatrix} \frac{h^2}{32}\varepsilon_x^0 \quad (87)$$

where

$$R_{1,1}^* = R_{1,1}^*(\varepsilon_x^0/2) \quad = \quad R_{1,2}^* = R_{1,2}^*(\varepsilon_x^0/2) \quad (88)$$
$$R_{2,1}^* = R_{2,1}^*(\varepsilon_x^0/2) \quad = \quad R_{2,2}^* = R_{2,2}^*(\varepsilon_x^0/2) \quad (89)$$
$$R_{12,1}^* = R_{12,1}^*(-\varepsilon_x^0) \quad \neq \quad R_{12,2}^* = R_{12,2}^*(\varepsilon_x^0) \quad (90)$$

With Eqs. (88)-(90) we obtain Eqs. (91) and (92):

$$\underline{f}^{0*} = \begin{bmatrix} N_x^* \\ N_y^* \\ N_{xy}^* \end{bmatrix} = \begin{bmatrix} 2(c_{11}+c_{12})R_{1,1}^*+2(c_{12}+c_{22})R_{2,1}^*+4c_{33}(R_{12,1}^*+R_{12,2}^*) \\ 2(c_{11}+c_{12})R_{1,1}^*+2(c_{12}+c_{22})R_{2,1}^*-4c_{33}(R_{12,1}^*+R_{12,2}^*) \\ 0 \end{bmatrix} \frac{h}{8}\varepsilon_x^0 \quad (91)$$

$$\underline{f}^{1*} = \begin{bmatrix} M_x^* \\ M_y^* \\ M_{xy}^* \end{bmatrix} = \begin{bmatrix} 4c_{33}(R_{12,2}^*-R_{12,1}^*) \\ -4c_{33}(R_{12,2}^*-R_{12,1}^*) \\ -2(c_{11}-c_{12})[R_{1,1}^*+R_{2,1}^*] \end{bmatrix} \frac{h^2}{32}\varepsilon_x^0 \quad (228)\ (92)$$

When every reliability function is approximately 1, Eqs. (91) and (92) become (*h*=1 mm):

$$\underline{f}^{0*} = \begin{bmatrix} c_{11}+2c_{12}+c_{22}+4c_{33} \\ c_{11}+2c_{12}+c_{22}-4c_{33} \\ 0 \end{bmatrix} \varepsilon_x^0 \frac{h}{4} = \begin{bmatrix} 182.60 \\ 146.20 \\ 0 \end{bmatrix} \varepsilon_x^0 \frac{h}{4} = \begin{bmatrix} 45.65 \\ 36.55 \\ 0 \end{bmatrix} \varepsilon_x^0 \quad (93)$$

$$\underline{f}^{1*} = -\begin{bmatrix} 0 \\ 0 \\ c_{11}-c_{12} \end{bmatrix} \varepsilon_x^0 \frac{h^2}{8} = -\begin{bmatrix} 0 \\ 0 \\ 145.96 \end{bmatrix} \varepsilon_x^0 \frac{h^2}{8} = -\begin{bmatrix} 0 \\ 0 \\ 18.25 \end{bmatrix} \varepsilon_x^0 \quad (94)$$

Figures 8 and 9 show the force and moment elements of $\underline{f}^{0*}$ and $\underline{f}^{1*}$ given by Eqs. (91) and (92), respectively, as functions of the strain load ε_x^0. The initial tangents (dashed lines) calculated with Eqs. (93) and (94) correspond to functioning without damage.

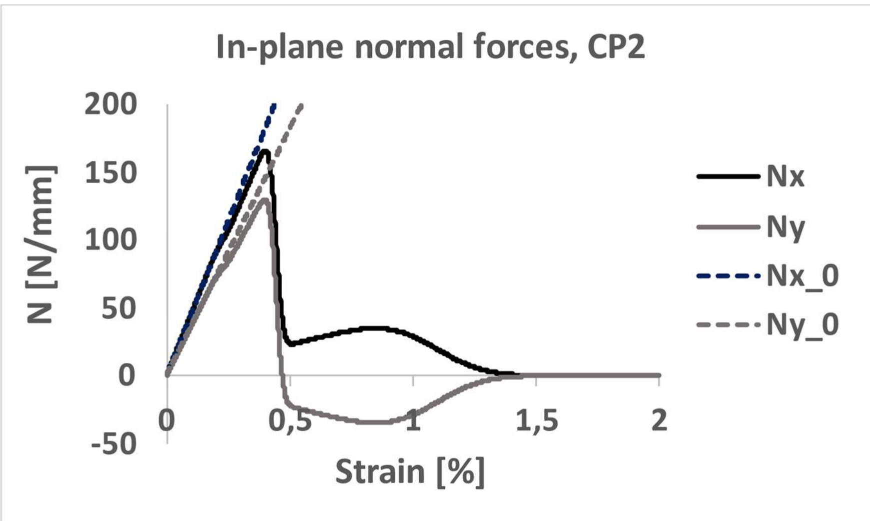
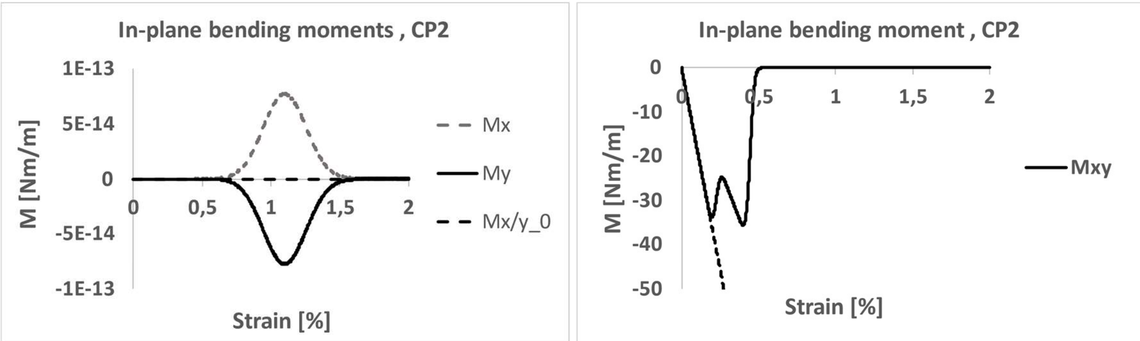


Figure 8 In-plane normal force functions N_x and N_y for specimen CP2

The difference in the shape of N_x and N_y (Figure 8) is caused by the different sign of the third term in Eq. (91). The moments M_x and M_y are very small and, according to Eq. (94), they are identically zero in the case of no damage (Figure 9). The two-peak shape of N_x, N_y and M_{xy} are considered damage-induced fluctuations.



(a) (b)

Figure 9 In-plane bending moment functions M_x and M_y (a), and M_{xy} (b) for specimen CP2

3.4 Modeling damage at the laminate level

When defining the reliability functions on the laminate levels, we utilized the statement based on Eqs. (48), (49) and (2) at the end of Chapter 2.2.5. According to that, the elements of the reliability matrix R_L^* defined on the laminate level may be approximated by different kind of mapping (e.g. the linear combination) of the MRFs R_1^*, R_2^*, and R_{12}^* defined on lamina levels.
On the other hand, the MRFs, R_{Lk}^* (k=1,...,6), as the elements of the reliability matrix R_L^*, can be estimated from mechanical tests performed on laminate specimens subjected to suitable loads. Without empirical data, of course, theory-based methods are used.
Thus, for the uniaxial tensile strain load, we chose and calculated two kinds of reliability functions, $R_{ZL}(u)$, defined on the laminate level, which are the approximations of the reliability function, $R_Z(u)$, defined on the lamina level where index Z denoted the in-plane force (Z=N) or moment (Z=M). They have the properties of the reliability functions, therefore they are positive and monotone decreasing, and their values are not greater than 1. Accordingly, the so-called safety-based approximation meets the next condition, that it is less than or equal to $R_Z(u)$ (fitting from below) but the mean linear error (MLE) is minimal:

$$R_{ZL}(u):\;\; R_{ZL}(u) \le R_Z(u), \quad MLE = \frac{1}{u_{max}} \int_0^{u_{max}} (R_Z(u) - R_{ZL}(u))du \to min \tag{95}$$

$$RMLE = \frac{MLE}{\max\limits_{0 \le u \le u_{max}} RZ(u)} = MLE \tag{96}$$

The safety-based approximation represents a kind of strictest criterion concerning the damage. Otherwise, the least squares–based approximation minimizes the mean squared error:

$$R_{ZL}(u):\;\; MSE = \frac{1}{u_{max}} \int_0^{u_{max}} [R_Z(u) - R_{ZL}(u)]^2 du \to min \tag{97}$$

In the latter case, we characterized the deviation with the relative mean squared error (RMSE):

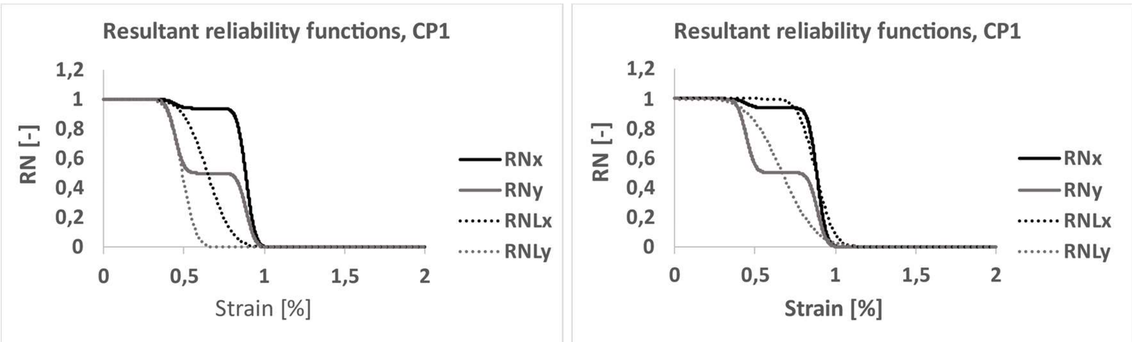
$$RMSE = \frac{\sqrt{MSE}}{\max\limits_{0 \le u \le u_{max}} RZ(u)} = \sqrt{MSE} \tag{98}$$

In this case, the maximum value of the reliability function is 1, hence RMLE and RMSE equal the MLE and the root of the MSE, respectively.
Figures 10 and 11 show the results of the calculations using the methods according to Eqs. (95) and (97). The strain parameters of the approximate reliability functions for the laminate level and those characterizing the goodness of fitting can be found in Table 5.

Parameters	**Safety based**			**Least squares based**		
	R_{NLx}	R_{NLy}	R_{MLx}	R_{NLx}	R_{NLy}	R_{MLx}
	ε_{NLx} [%]	ε_{NLy} [%]	ε_{MLx} [%]	ε_{NLx} [%]	ε_{NLy} [%]	ε_{MLx} [%]
Mean [%]	0.650	0.490	0.889	0.880	0.670	0.890
CV [-]	0.185	0.140	0.050	0.100	0.250	0.050
SD [%]	0.120	0.069	0.044	0.088	0.168	0.045
RMLE [%]	10.63	8.84	0.24	-	-	-
RMSE [%]	-	-	-	4.14	10.02	0.88

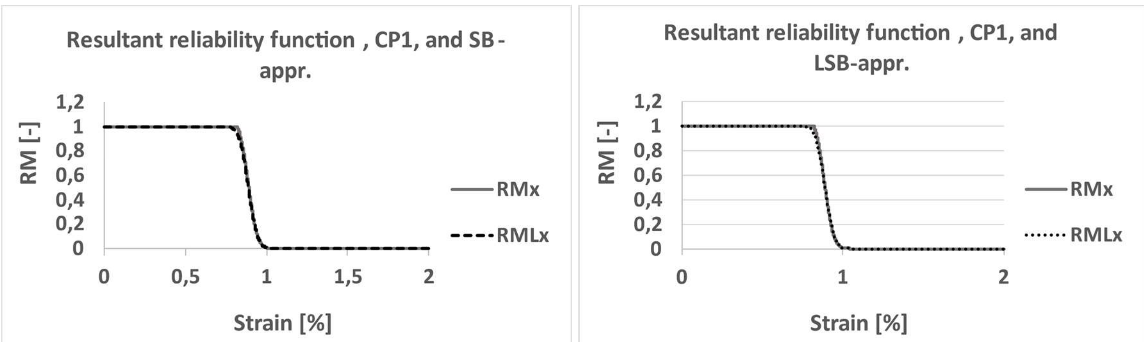
Table 5 Statistical parameters of the breaking strains defined on the laminate level

In the case of the in-plane forces, the approximate reliability functions on the laminate level have remarkable deviations compared to those on the lamina (Figure 10) while the deviations are very small for the reliability functions belonging to the in-plane moments (Figure 11).



(a) (b)

Figure 10 Force reliability with damage on the lamina level (RNx and RNy), and safety- (a) and least square based–reliability (b) on laminate level (RNLx and RNLy)



(a) (b)

Figure 11 Moment reliability with damage on lamina (RMx) and its safety-based (a) and least square–based (b) approximations on the laminate level (RMLx)

The safety-based and the least square–based approximations in Figure 10 strongly differ, while those in Figure 11 essentially cover each other.

In the case of the uniaxial tensile test in the diagonal direction, the approximate reliability function on the laminate level can be determined similarly to the above way.

4 SUMMARY AND CONCLUSIONS

Based on the stochastic material law, we derived a possible approach of the stochastic version of the Classical Composite Laminate Theory (SCLT) for strain load. There are different angles between the main directions of the laminas. By transforming the general Hooke's law fixed to the local coordinate system of the lamina into a common global coordinate system, we gave angle dependence to the elements of the stiffness matrix establishing a relationship between the stochastic processes of deformation and stresses. The plane forces and moments are also stochastic processes.
The expected evolution and standard deviation of edge forces and moments occurring during deformation-driven stochastic processes were also given.
All the above can be completed by determining the standard deviation and the confidence interval, giving a possibility for designers and engineers to design not only the construction or machine parts but also their reliability through taking into account the damage and failure processes as well.
To show the applicability of the SCLT, we demonstrated it by tensile testing a simple two-ply composite sheet specimen in two different directions and modeling the damage processes on lamina and laminate levels. From the example, we conclude that the application of the E-bundle–based memory reliability functions make it possible to predict and analyze the damage and failure processes of the laminate. Applying them on the lamina level reveals the effect of local damage on the stress components including damage-induced stress peaks as well.

Acknowledgements

This work was supported by Hungarian National Research, Development and Innovation (NRDI) Office through grant OTKA K 116189. The research reported in this paper is part of project no. TKP-6-6/PALY-2021 has been implemented with the support provided by the Ministry of Culture and Innovation of Hungary from the NRDI Fund, financed under the TKP2021-NVA funding scheme.
The authors would like to thank Prof. Dr. Imre Bojtár, Budapest University of Technology and Economics, for his helpful comments and suggestions on the manuscript.

REFERENCES

1. Reddy JN. *Mechanics of Laminated Composite Plates and Shells – Theory and Analysis*. 2nd ed. Boca Raton – London – New York: CRC Press, 2003.

2. Kollár L and Springer GS. *Mechanics of Composite Materials*. Cambridge UK: Cambridge University Press, 2003.

3. Gibson RF. *Principles of Composite Material Mechanics*. 4th ed. Boca Raton, FL, USA: CRC Press, 2016.

4. Luo RK. Effective strain criterion under multimode and multiaxial loadings – A rubber S-N curve with the scatter-band factor of 1.6 from 90 fatigue cases. *Express Polym Lett* 2022; 16/2: 130-141.

5. Paris F. A Study of Failure Criteria of Fibrous Composite Materials. Final report, NASA/CR-2001-210661, 2001. https://ntrs.nasa.gov/archive/.../20010035883.pdf

6. Nordmann J, Naumenko K, and Altenbach H. Cohesive Zone Models—Theory, Numerics and Usage in High-Temperature Applications to Describe Cracking and Delamination. In: Naumenko K and Krüger

M (eds) *Advances in Mechanics of High Temperature Materials*. London-Berlin-New York: Springer Nature Switzerland AG, 2020, pp.131-168. https://doi.org/10.1007/978-3-030-23869-8

7. Elices M, Guinea GV, Gómez J, et al. The cohesive zone model: advantages, limitations, and challenges. *Eng Fract Mech* 2002; 69: 137-163.

8. Sutherland LS and Guedes Soares G. Review of Probabilistic Models of the Strength of Composite Materials. *Reliab Eng Syst Safe* 1997; 56: 183-196.

9. Hader A, Achik I, Lahyani A, et al. Characterization of the failure process in composite materials by the Fiber Bundle Model. *Superlattice Microst* 2014; 71: 30-37.

10. Hansen A, Hemmer PCh and Pradhan S. The Fiber Bundle Model: Modeling Failure in Materials. Weinheim, Germany: Wiley-VCH Verlag GmbH, 2015.

11. Vas LM and Tamás P. Modelling Method Based on Idealised Fiber Bundles. *Plast Rubber Compos* 2008; 37(5/6): 233-239.

12. Vas LM and Rácz Zs. Modeling and Testing the Fracture Process of Impregnated Carbon Fiber Roving Specimens During Bending Part I. Fiber Bundle Model. *J Compos Mater* 2004; 38/20: 1757-1785.

13. Vas LM, Kocsis Z, Czigány T, et al. Novel Evaluation Method Of Acoustic Emission Data Based On Statistical Fiber Bundle Cells. *J Compos Mater* 2019; 53/17: 2429–2446.

14. Vas LM, Tamás P, Bognár E, Nagy P, et al. Nonlinear Fiber-Bundle-Cells Based Modeling of Human Tissue Samples. *Biomech Model Mechan* 26 October 2022; 21/6: 1803-1823. DOI:10.1007/s10237-022-01621-1.

15. Vas L.M., Tamás P.: Developing a Stochastic Linear Elastic Material Model and Composite Laminate Theory Including Failures, Using the Fiber Bundle-Based Approach, Part I: Stochastic Linear Elastic Material Model (submitted together with this Part II)