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DEVELOPING A STOCHASTIC LINEAR ELASTIC MATERIAL MODEL AND COMPOSITE LAMINATE THEORY INCLUDING FAILURES, USING THE FIBER BUNDLE–BASED APPROACH

Part I: Stochastic Linear Elastic Material Model

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Notation

Abbreviations
CV – Coefficient of Variation
DMR – Duplex Memory Reliability (function, tensor, matrix, …)
FBC – Fiber Bundle Cells
FEM – Finite Element Method
GMS – Global Memory Stiffness
PDF – Probability Distribution Function
RF – Reliability Function
SD – Standard deviation
WF – Window Function

General signs and notations
a, *A* – cursive lowercase or uppercase letters symbolize real parameters, variables, or functions
$\underline{\mathrm{a}}$ – upright lowercase underlined letters symbolize vectors
A, A – upright bold and regular uppercase letters symbolize tensor and matrix, respectively
B – general index for normal compressive (-*B*=*C*) or tensile (*B*=*T*) or shear (±*B*=±*S*) breaking strain
$\mathbb{D}$(X) – Standard Deviation of the stochastic variable X
$\mathbb{E}$(X) – Expected value of the stochastic variable X
$\mathbb{P}$(A) – Probability of event A
$P_X(u)$= $\mathbb{P}$(X<u) – Probability Distribution Function (PDF) of the stochastic variable X
$\widetilde{\mathrm{Y}}$, $\tilde{Y}$– over-tilde~ designates the stochastic process character of matrix (Y) or composite function (Y) containing empirical duplex RFs (e.g. $r_{CT}(u)$ or $r_{CT}^*(u)$) as entries or internal variables, respectively
$\mathrm{Y}^*, \widetilde{\mathrm{Y}}^*$ – superscript asterisk* denotes the memory property of window functions (χ_{Zn}^*), empirical ($r_X^*(u)$) and expected reliability functions ($R_X^*(u)$), or stochastic vector or matrix (e.g. $\tilde{\sigma}^*, \widetilde{\mathrm{C}}^*, \widetilde{\mathrm{R}}^*$). The expectation of the latter expressions has the memory property as well, therefore it is indicated by the asterisk only (e.g. $\sigma^*, \mathrm{C}^*, \mathrm{R}^*$).
□ – at the end of a proof, it indicates completion (Q.e.d.)

Variables and parameters
f, f_S – normal and shear engineering stress, respectively
$\underline{\mathrm{f}}^0$, $\underline{\mathrm{f}}^1$ – in-plane force and moment vectors, respectively
$g(u)$, $g_S(w)$ – tensile and shear force–strain characteristic curves of the intact fibers, respectively

h – thickness of the laminate
$r_X(u)$, $r_Y(w)$ – empirical RF of the fiber bundle related to $X\in\{C,T\}$ and $Y\in\{-S,S\}$, respectively
$r_{CT}(u)$, $r_{-S,S}(w)$ – empirical duplex RFs of the fiber bundle, respectively
u, w – normal and shear strain load of the E-bundle, respectively
$u_m(t)$, $w_m(t)$ – minimum of the normal and shear strain load in the time interval $[0,t]$, respectively
$u_M(t)$, $w_M(t)$ – Maximum of the normal and shear strain load in the time interval $[0,t]$, respectively
A, A_0 – cross-sectional area of the loaded and unloaded fibers, respectively
A, B, W – general stiffness minor matrices of the laminate
$\mathbf{C}$ – stiffness tensor
C = $[c_{ij}]$ – compacted matrix of the stiffness tensor $\mathbf{C}$ (constant elements)
$\widetilde{C}^*= C\widetilde{R}^*$, $C^*=CR^*$ – empirical and expected memory stiffness matrix
$\mathbf{D}$ – strain tensor for the lamina
D = $[\varepsilon_{ij}]$ – compacted matrix of the strain tensor $\mathbf{D}$ and the normal ($i=j$) or shear ($i\neq j$) strain elements
E_n and E – tensile elastic modulus of the n^{th} fiber and its expectation
$F_n(u)$, $F_{Sn}(w)$ – tensile and shear force arising in the n^{th} fiber
$F(u)$, $F_S(w)$ – tensile and shear force arising in the fiber bundle
G_n and G – shear elastic modulus of the n^{th} fiber and its expectation
L, L_0 – length of the loaded and unloaded fiber bundle, respectively
M – number of mechanical measurements or observations
N – number of fibers in the E-bundles
$R_X(u)$, $R_Y(w)$ – expected RF of the fiber bundle related to $X\in\{C,T\}$ and $Y\in\{-S,S\}$, respectively
$R_{CT}(u)$, $R_{-S,S}(w)$ – expected duplex RFs of the fiber bundle, respectively
$\widetilde{\boldsymbol{R}}^*$, $\mathbf{R}^*$ – empirical and expected DMR tensors, respectively
$\widetilde{R}^* = [r^*_{ij}]$, $R^* = [R^*_{ij}]$ – compacted matrix of the empirical and expected DMR tensors
S_0 – shear surface area of the fibers
$\mathbf{S}$ – stress tensor for the lamina
S = $[\sigma_{ij}]$ – matrix of the stress tensor $\mathbf{S}$ and the normal ($i=j$) or shear ($i\neq j$) stress elements
U = $[u_{ij}]$ – strain load matrix and its elements related to the $(ij)^{th}$ fiber bundle ($u=u_{ii}$; $w=u_{ij}$, $i\neq j$)
γ – shear strain
ε – normal strain
$-\varepsilon_{-Bn}$, ε_{Bn} – normal ($-B=C$, $B=T$) and shear ($B=S$) breaking strain of the n^{th} fiber, respectively
$\underline{\varepsilon} = [\varepsilon_i]$ – strain vector
$\kappa_{CT}(u)$, $\kappa_{S,S}(w)$ – mean normal and shear modulus functions of the fiber bundle, respectively
κ^0_x, κ^0_x, κ^0_{xy} – out-of-plane deflection and torsion deformations
ν_{ij} – Poisson's coefficient of the fibers
σ – normal stress
$\underline{\sigma} = [\sigma_i]$ – stress vector
τ – shear stress
χ_{Zn}, χ^*_{Zn} – WF and MWF of the n^{th} fiber related to strain load $Z\in\{C, T, -S, S\}$, respectively
$\chi_{-B,Bn}$, $\chi^*_{-B,Bn}$ – duplex WF and MWF of the n^{th} fiber related to $-B$, $B\in\{CT; -S, S\}$, respectively
$\psi(u)$ – contraction function for the fibers

Abstract

Based on the fiber bundle cells (FBC) theory, we introduced the so-called memory reliability functions to take into account the different changing strain loads including both the monotone, the pulsating and the alternating modes. We proved that the duplex compressive–tensile memory reliability function is the product of the compressive and the tensile memory reliability functions. Utilizing the generalized Hooke's law and the memory reliability functions, we developed a stochastic linear elastic material law that also represents the damage and failure processes with normal and shear strain loads.

This made it possible to calculate the expected value as the variance of the stress response process and determine its confidence interval at any strain load.

Keywords: material law, composites, fiber bundle, memory reliability function, stochastic modeling

1 INTRODUCTION

In the last few decades, fiber-reinforced composites have become very important structural materials and they have been widely used in both engineering constructions and everyday life. All that is especially true for composites with a polymer matrix because of their advantages, such as low weight, high stiffness and strength, easy production even in large sizes, and relatively low cost. Designing for engineering or construction, including machine parts made from high-performance composite materials needs suitable and accurate mathematical models and calculation methods for parts with optimized geometry, mass, strength properties, energy consumption, and reliability-based life expectancy.[1]

In general, the material models used for designing parts assume a linear elastic mechanical behavior based on Hooke's Law[1-5].

With Finite Element Method (FEM)–based simulation models, complex engineering constructions can be designed, where, in general, the load is assumed to be below the load causing failure. Correct design should be based on elasticity and strength data obtained from mechanical tests of different types and some of the failure criteria[1-3, 6-8]. According to a NASA report in 2001[7], in practice and in general, researchers or engineers apply a set of simple criteria related to the damage or failure types rather than the usual failure criteria of global and average view when they design constructions of high standard.

Damage makes the stress–strain relationships non-linear and stochastic. To take into account the non-linearity effect, for example, the so-called Cohesive Zone Method (CZM) has been developed in fracture mechanics for modeling crack propagation and the delamination process in solids[8, 9]. Its simplified form called the bi-linear CZM model, where the traction-separating law has a triangle shape, has often been applied to describe failure phenomena in finite element (FEM) simulations[8-12].

However, considering the stochastic nature of the damage and failure processes in structural mechanics[13] would require a probabilistic material law and determining the confidence interval or range at a given probability level. One of the most effective methods to take into account stochastic effects of several types of damage is the fiber bundle model[14-22], which has widely been used for describing failure processes of various kinds occurring in different materials[23-28]. The theory and some methods of using the classical fiber bundle were developed essentially in the period between 1920 and 1990[14-20] by Peirce[14], Daniels[15], Phoenix[16-18], and Harlow[17, 19], among others. The classical fiber bundle[19] consists of elastic and straight (polymer) fibers with the same tensile modulus. The fibers have the same linear density and are parallel to the uniaxial tensile load, and their breaking strain is a stochastic variable of the same probability distribution. In this period, the classical fiber bundle–based method was applied to predicting the expectation and/or the distributions of the strength of fibrous structures[19] and the so-called size effects in strength[20], and to modeling how stress concentration develops under different conditions[20-22]. Moreover, Curiskis and Carnaby[23] proposed a method to treat 3D fiber bundles with continuum mechanics. Some recent applications used the fiber bundle model in FEM simulations[24-28]. Moreover, the fiber bundle model gained remarkable importance in modeling the mechanical behavior of metals[27].

To describe the total deformation and damage process up to ultimate failure and take into consideration statistical defects in other mechanical, structural, and geometrical properties, we have developed the so-called Fiber Bundle Cells (FBC) modeling method, which defines several fiber bundles of different properties, where the fibers are linear or nonlinear elastic (E – elastic)[29, 30]. The simple E-bundle is similar to the classical one, when the fibers are linear elastic, but their tensile stiffness may be stochastically variable. Spoiling the ideal properties of the E-bundle provides other fiber bundles representing further statistical defects such as crimped (EH-bundle) or oblique (ET-bundle) fibers or the slippage of fibers out of their gripping realized by their vicinity (ES-bundle)[29, 30]. They are called fiber bundle cells (FBCs). In general, the FBC model is a network of fiber bundle cells weighted by their fiber number fractions. The FBC modeling method can be used in two different ways[29-40].

One of them is the structural–mechanical modeling of mechanical behavior at fiber/matrix level during different mechanical tests where the FBC model plays the role of the material law[29]. Such are for example, the tensile testing of a twisted filament fiber bundle[31] or a unidirectional short fiber-reinforced composite[32, 33], or the 3-point bending of a unidirectional carbon/epoxy composite beam[34, 35].

The other is the phenomenological modeling of measured stress–strain relationships by connecting weighted FBCs in parallel. In this case, the FBC model is fitted to the test and provides a decomposition of the measured stress–strain curve, which supplements the usual test results with important additional information[29, 30]. For instance, the fitted linear FBC model of an electrospun nanofibrous web made it possible to assess the strength of the single nanofibers indirectly, without any measurements on single nanofibers[36]. Since the weighted parallel connection of nonlinear E-bundles can approximate any FBC, it can be used for any other fiber bundle cell, that is, any FBC model as well[37-40]. This fact is utilized when acoustic emission and stress–strain behavior are modeled simultaneously during the tensile testing of short glass fiber–reinforced PP composite plates. This way, the nonlinear E-bundles represent different failure modes, such as micro-cracking, fiber/matrix debonding, or the slippage or breakage of fibers[37]. Other applications were modeling the tensile stress–strain relationships of woven fabrics and their composites[38] or describing the rugged tensile stress–strain curve of single tests performed on human tissue specimens[39]. It could also be applied to modeling the mechanical behavior of the microstructural phases in nanotube- and microfiber-reinforced thermoplastic matrix composites[40].

Based on the results and experiences we have gained by FBC modeling the damage and failure processes in different types of fibrous structures and materials, we aimed to develop a 3D fiber bundle–based material law. Hence, in this paper, we propose an FBC-based material model that is linear elastic below destructive load level and can describe the nonlinear damage and failure processes of different types occurring at destructive load levels.

2 LINEAR E-BUNDLES

In this paper, we created a statistical material model for linear elastic materials, which also describes the damage and failure processes besides deformation behavior. It is based on special E-bundles in which the idealized model fibers are unidirectional and have finite tensile, compressive, and shear moduli, which may be stochastic parameters. Hence, as a sort of generalization, these E-bundle model-fibers can be stretched, compressed and sheared, and the related stress–strain relationships are linear elastic. Otherwise, these model fibers fail (break, fracture, or rupture) when their normal (tensile or compressive) or shear strain reaches a random value called breaking strain, which has a known distribution function. In the case of the originally introduced linear E-bundle [29, 30], the strain load as stimulus was described with a monotone increasing tensile load–time function which usually is not true in practice. Therefore, subsequently, we try to extend the usability of linear E-bundles for alternating tensile and compressive, and shearing load.

2.1 Basic properties of the FBC material model

Consider a small cubic part of a continuum material exhibiting linear elastic behavior given by Hooke's law. With a small load, the mechanical behavior of this cube is the same as that of the original material. To model its behavior in the region of the damage and failure, the small cubic-shaped material volume is considered fiberized into virtual sticklike fibers. They become independent discrete elements and create an E-bundle. The abstract form of the E-bundle consists of virtual fibers—straight lines of volume-less mass points. In every main direction, different E-bundles characterize the mechanical properties of the material (Figures 1.e and 1.f). Accordingly, the sticklike fibers are built up of elementary cubic parts with the same general elastic behavior as the material in their vicinity (Figure 1).

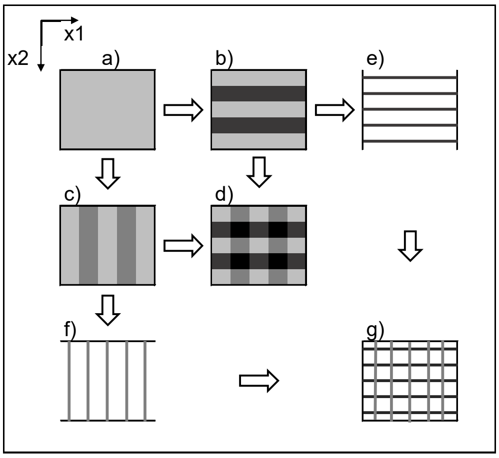


Figure 1 Virtual partitioning of a small 2D square volume (a) in two directions into 5 parallel sticks (b, c) then 5 model fibers (e, f) and their union (d; g)

Thus, the fibers are discrete sticklike continuum objects (Figure 1) having the same unloaded cross section (A_0) and creating a linear E-bundle. This means that they are:

- linear elastic, that is, their stress–strain relationships are homogeneous linear functions for any load mode (compression, tensioning/stretching, shearing),
- straight and parallel to each other,
- not pre-strained or loose,
- ideally gripped, that is, they do not fail in the grips and are not pulled out of the grips,
- independent of each other, that is, there is no mechanical contact or interaction between them.

The number of fibers in a unidirectional E-bundle is finite and denoted by N, hence the cross-sectional area of the bundle is NA_0. In every case, the fiber bundle is subjected to controlled time-dependent (t) strain deformation as stimulus, which is denoted by $u(t)$ or $w(t)$ when normal or shear strain is used, respectively.

The normal bundle strain, u, is parallel to the fibers while the shear bundle strain, w, is perpendicular to them:

$$u = \frac{\Delta L}{L_0} = \frac{L - L_0}{L_0} \quad (1)$$

$$w = \frac{\Delta H}{L_0} = \tan(\gamma) \approx \gamma \quad (2)$$

where L_0 and L are the lengths of the unloaded and loaded fiber bundle, respectively, while ΔL and ΔH are the normal and shear displacements of the loaded bundle, respectively. The normal (ε) and shear (γ) strains of each fiber are assumed identical with those of the fiber bundle

$$\varepsilon(u) = u \quad (3)$$

$$\gamma(w) = \arctan(w) \approx w \quad (4)$$

Considering a Cartesian coordinate system with axes x_1, x_2 and x_3, and deformation matrices $U=[u_{ij}]=[\varepsilon_{ij}]=D$, where $u \in \{u_{11}, u_{22}, u_{33}\}$ and $\varepsilon \in \{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}\}$ denote some of the normal bundle and fiber strains while $w \in \{u_{ij}: i \neq j\}$ and $\gamma \in \{\varepsilon_{ij}: i \neq j\}$ denote some of the shear bundle and fiber strains. For example, in a plane, $u=u_{11}$ is parallel to the coordinate axis x_1, consequently w is parallel to axis x_2 or x_3.

2.2 Mechanical properties of single model fibers

Let us suppose that the normal bundle strain, u (or $u_1=u_{11}$) is parallel to the axis x (or x_1) and so are the fibers considered.

2.2.1 Nondestructive monotonic load – Deterministic mechanical behavior

Stress–strain relationships

The compressive ($u\leq 0$) and tensile ($0\leq u$) stress–strain relationships of the n^{th} fiber ($n=1,\ldots,N$) are the same, and the compressive and tensile moduli are equal and denoted by E_n. For the n^{th} fiber, the next characteristic function, $g_n(u)$, describes the real normal stress response of the fibers when no damage or failure occurs during loading:

$$\sigma(u) = g_n(u) = E_n\varepsilon(u) = E_n u \tag{5}$$

The negative ($w\leq 0$) and positive ($0\leq w$) shear stress–strain relationships of the n^{th} fiber are the same, that is, the negative and positive shear moduli are equal (denoted by G_n). For the n^{th} fiber, the characteristic function, $g_{Sn}(w)$, describes the shear stress response (τ) of the fibers when no damage or failure occurs during loading:

$$\tau(w) = g_{Sn}(w) = G_n\gamma(w) = G_n w \tag{6}$$

Poisson effects

The fibers as continuum sticks exhibit the Poisson effect, that is, a contraction in the cross section, when they are subjected to normal strain load. On the other hand, the shear strain load of the bundle is assumed to result in volume-preserving shear deformation of the fibers, although without any Poisson's effect (Figure 2).

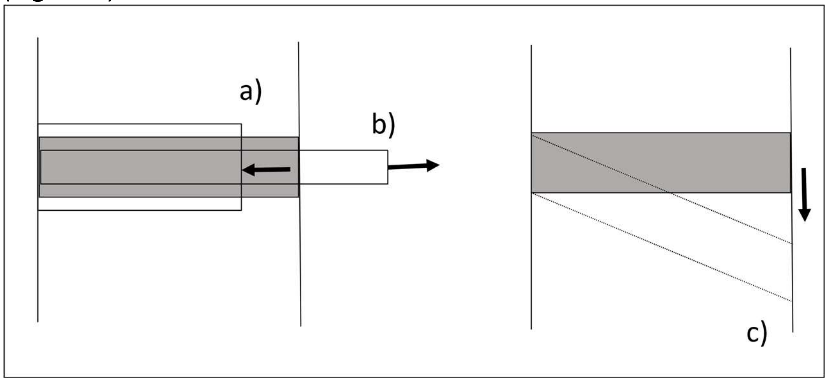


Figure 2 Assumed mechanical behavior of the fibers as elastic continuum sticks (the blue-filled rectangles represent the unloaded fibers) when subjected to compressive (a), tensile (b), and shear (c) strain loads

For example, when tensile strain load is acting in direction x_1, $u=u_1$, the crosswise contraction of the n^{th} fiber, as the Poisson effect, can be described with an *area contraction function*, $\psi(u):=\psi_1(u)$, which is the ratio of the loaded, $A:=A_1$, and the initial, $A_0:=A_{10}$, cross section areas of the fiber ($0\leq u$) (Figure 1):

$$\Psi(u) = \frac{1}{1+u} \leq \psi(u) = \frac{A(u)}{A_0} = \frac{A_1(u)}{A_{10}} = \psi_1(u) \leq 1 \tag{7}$$

where $\Psi(u)=\psi(u)$ is valid for volume constancy, while $\psi(u)=\psi_1(u)\equiv 1$ is true for cross section constancy. When the strain load in direction x_1 is small, the *linear contraction functions*, $\psi_{12}(u)$ and $\psi_{13}(u)$ related to the two crosswise directions (2 and 3) and given by the Poisson's coefficients, ν_{12} and ν_{13} [1-3] are used for linear elastic materials:

$$\psi_1(u) = \psi_{12}(u)\psi_{13}(u) = (1-\nu_{12}u)(1-\nu_{13}u) \approx 1-(\nu_{12}+\nu_{13})u, \qquad 0\leq u\leq u_V \tag{8}$$

For example, the upper limit, u_V, can be defined by solving the following equation:

$$u_V:\ \frac{1}{1+u} = 1-(\nu_{12}+\nu_{13})u \quad \Rightarrow \quad u_V = \frac{1-(\nu_{12}+\nu_{13})}{\nu_{12}+\nu_{13}} \tag{9}$$

At the value u_V, linear contraction is positive and reaches the value given by volume constancy ($0<\psi_1(u_V)=\Psi(u_V)$). The contraction function given by Eq. (7) can be applied to a compressive load ($u\leq 0$), in the strain load range as well (-1, 0]:

$$\Psi(u) = \frac{1}{1+u} \geq \psi(u) = \frac{A(u)}{A_0} \geq 1 \tag{10}$$

Fiber forces

The compressive ($u\leq 0$) or tensile ($0\leq u$) force arising in the n^{th} fiber can be calculated with Eqs. (5) and (7) when no damage or failure occurs during loading:

$$F_n(u) = A(u)\sigma_n(u) = A_0\psi(u)\sigma_n(u) = E_nA_0\psi(u)\varepsilon(u) = E_nA_0\psi(u)u \tag{11}$$

Correspondingly, engineering stress, f_n, can be defined as a specific force that is larger or smaller than the real compressive or tensile stress, σ_n, respectively:

$$f_n(u) = \frac{F_n(u)}{A_0} = \psi(u)\sigma_n(u) = E_n\psi(u)\varepsilon(u) = E_n\psi(u)u = \begin{cases} \geq \sigma_n(u), & u \leq 0 \\ \leq \sigma_n(u), & 0 \leq u \end{cases} \tag{12}$$

The negative ($w\leq 0$) and positive ($0\leq w$) shear force arising in the n^{th} fiber can be calculated with Eq. (6) and the constant shear surface S_0 when no damage or failure occurs during loading:

$$F_{Sn}(w) = S_0\tau_n(w) = G_nS_0\gamma(w) = G_nS_0w \tag{13}$$

Consequently, engineering shear stress equals real stress:

$$f_{Sn}(w) = \frac{F_{Sn}(w)}{S_0} = \tau_n(w) = G_nw \tag{14}$$

2.2.2 Destructive monotonic load – Stochastic mechanical behavior

Breaking strains and the window functions

When the compressive or tensile strain load is increased, u, the n^{th} fiber (n=1, 2,…, N) fractures at a random compressive breaking strain, $-\varepsilon_{Cn}$, ($u\leq 0$) without any elastic buckling or breaks at a random tensile strain, ε_{Tn}, ($0\leq u$). Hence, as far as the fiber is intact, normal strain remains between these limits:

$$-\varepsilon_{Cn} \leq \varepsilon(u) = u \leq \varepsilon_{Tn} \tag{15}$$

E_n, ε_{Cn}, and ε_{Tn} are independent stochastic variables with probability distribution functions $P_E(x)$ ($0\leq x$), $P_{\varepsilon C}(u)$ and $P_{\varepsilon T}(u)$ ($-\infty<u<\infty$), respectively, which are the same for every fiber and have finite expected values and variances.

The n^{th} fiber (n=1, 2,…, N) fractures at a random negative shear breaking strain, $-\varepsilon_{-Sn}$, ($w\leq 0$) or breaks at a random positive shear strain, ε_{Sn}, ($0\leq w$). Hence, as far as the fiber is intact, shear strain remains between these limits:

$$-\varepsilon_{-Sn} \leq \gamma(w) = w \leq \varepsilon_{Sn} \tag{16}$$

G_n, ε_{-Sn}, and ε_{Sn} are independent stochastic variables with probability distribution functions $P_G(x)$ ($0 \leq x$), $P_{-\varepsilon S}(w)$ and $P_{\varepsilon S}(w)$ ($-\infty < w < \infty$), respectively, which are the same for every fiber and have finite expected values and variances.
In addition, all the above-mentioned stochastic parameters, such as the compressive, tensile, and shear parameters are independent of each other as well.

Definition

•For compressive and tensile monotonic strain loads, the following so-called ***window functions*** (WF), χ_{Cn} and χ_{Tn} are defined for the n^{th} fiber, whose value is 1 while there is no failure and 0 at or after failure:

$$\chi_{Cn}(u) = \chi_C(u, -\varepsilon_{Cn}) = \begin{cases} 1, & -\varepsilon_{Cn} < u < \infty \\ 0, & \infty < u \leq -\varepsilon_{Cn} \end{cases} \quad (17)$$

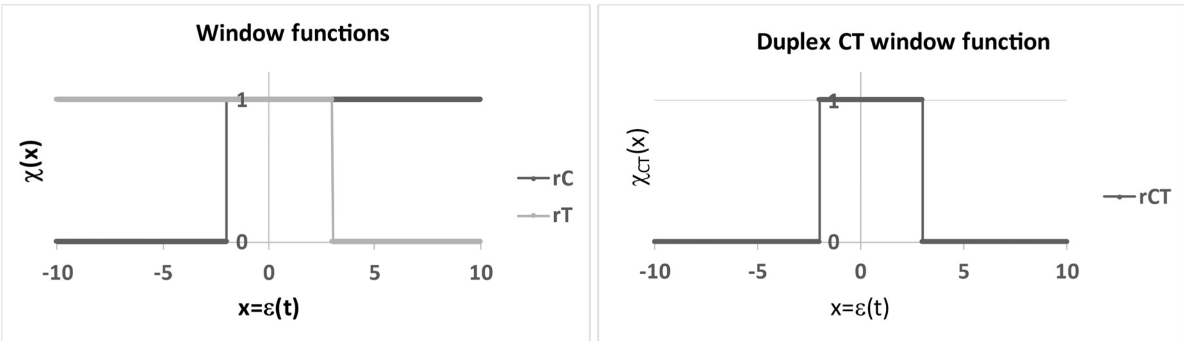
$$\chi_{Tn}(u) = \chi_T(u, \varepsilon_{Tn}) = \begin{cases} 1, & -\infty < u < \varepsilon_{Tn} \\ 0, & \varepsilon_{Tn} \leq u < \infty \end{cases} \quad (18)$$

•Similarly, for negative and positive monotonic shear strain loads, the following so-called ***shear window functions*** (shear WF), χ_{-Sn} and χ_{Sn} are defined for the n^{th} fiber:

$$\chi_{-Sn}(w) = \chi_{-S}(w, -\varepsilon_{-Sn}) = \begin{cases} 1, & -\varepsilon_{-Sn} < w < \infty \\ 0, & -\infty < w \leq -\varepsilon_{-Sn} \end{cases} \quad (19)$$

$$\chi_{Sn}(w) = \chi_S(w, \varepsilon_{Sn}) = \begin{cases} 1, & -\infty < w < \varepsilon_{Sn} \\ 0, & \varepsilon_{Sn} \leq w < \infty \end{cases} \quad (20)$$

It is assumed that the fracture or breakage is complete and ultimate, so the fractured or broken fiber will not transmit any stress or force.
Figure 3.a shows an example of the compressive and tensile window functions of a single fiber, which can be regarded as a one-fiber bundle.



(a) (b)

Figure 3 Example of the compressive and tensile window functions of a single fiber (a) and the normal duplex window function (b)

Definition

•For both compressive and tensile monotonic strain loads, the ***normal duplex window function*** (n. duplex WF), χ_{CTn} can be defined for the n^{th} fiber. Its value is 1 while there is no failure and 0 at or after failure (Figure 3.b):

$$\chi_{CTn}(u) = \chi_{CT}(u, -\varepsilon_{Cn}, \varepsilon_{Tn}) = \begin{cases} 1, & -\varepsilon_{Cn} < u < \varepsilon_{Tn} \\ 0, & u \leq -\varepsilon_{Cn} \ or \ \varepsilon_{Tn} \leq u \end{cases} \quad (21)$$

•Similarly, for both negative and positive monotonic shear strain loads, the negative–positive ***shear duplex window function*** (s. duplex WF), $\chi_{-S,Sn}$, is defined for the n^{th} fiber as the following:

$$\chi_{-S,Sn}(w) = \chi_{-S,S}(w, -\varepsilon_{-Sn}, \varepsilon_{Sn}) = \begin{cases} 1, & -\varepsilon_{-S} < w < \varepsilon_{Sn} \\ 0, & w \leq -\varepsilon_{-Sn} \ or \ \varepsilon_{Sn} \leq w \end{cases} \quad (22)$$

Statement:
For compressive and tensile monotonic normal strain loads, the normal duplex WF, χ_{CTn}, can be calculated as the product of χ_{Cn} and χ_{Tn} :

$$\chi_{CTn}(u) = \chi_{CT}(u, -\varepsilon_{Cn}, \varepsilon_{Tn}) = \chi_C(u, -\varepsilon_{Cn})\chi_T(u, \varepsilon_{Tn}) = \chi_{Cn}(u)\chi_{Tn}(u) \tag{23}$$

For negative and positive monotonic shear strain loads the shear duplex WF, $\chi_{-S,Sn}$, can be calculated as the product of χ_{-Sn} and χ_{Sn} :

$$\chi_{-S,Sn}(w) = \chi_{-S,S}(u, -\varepsilon_{-Sn}, \varepsilon_{Sn}) = \chi_{-S}(u, -\varepsilon_{-Sn})\chi_S(u, \varepsilon_{Sn}) = \chi_{-Sn}(w)\chi_{Sn}(w) \tag{24}$$

Proof: Considering the definitions given by Equations (21) and (22), the statements are trivial. For example, the expressions in Equations (21) and (23) are equal for every value of *u*, that is, according to the definitions given by Equations (17) and (18), the product equals 1 if and only if both factors are 1 and is 0, otherwise:

$$\chi_{Cn}(u)\chi_{Tn}(u) = \chi_C(u, -\varepsilon_{Cn})\chi_T(u, \varepsilon_{Tn}) = \begin{cases} 1, & -\varepsilon_{Cn} < u \ \ and \ \ u < \varepsilon_{Tn} \\ 0, & u \leq -\varepsilon_{Cn} \ \ or \ \ \varepsilon_{Tn} \leq u \end{cases} = \chi_{CTn}(u)$$

Eq. (24) can be seen similarly as that above. ☐
Figure 3.b demonstrates that the normal duplex WF of a single fiber can be regarded as the product of the compressive and tensile window functions (Figure 3.a).

Stress–strain and force–strain relationships including failures

When damage or failure may occur during monotonic normal strain loading, the stress arising in the n^{th} fiber can be calculated for a compressive ($u \leq 0$) and tensile ($0 \leq u$) load with Eqs. (5), (21), and (23):

$$\sigma_n(u) = g_n(u)\chi_{CTn}(u) = E_n u \chi_{CT}(u, -\varepsilon_{Cn}, \varepsilon_{Tn}) = E_n u \chi_{Cn}(u)\chi_{Tn}(u) \tag{25}$$

Similarly, with the use of Eqs. (6), (22), and (24), the shear stresses for monotonic negative ($w \leq 0$) and positive ($0 \leq w$) shear strain load are given by:

$$\tau_n(w) = g_{Sn}(w)\chi_{-S,Sn}(w) = G_n w \chi_{-S,S}(u, -\varepsilon_{-Sn}, \varepsilon_{Sn}) = G_n w \chi_{-Sn}(w)\chi_{Sn}(w) \tag{26}$$

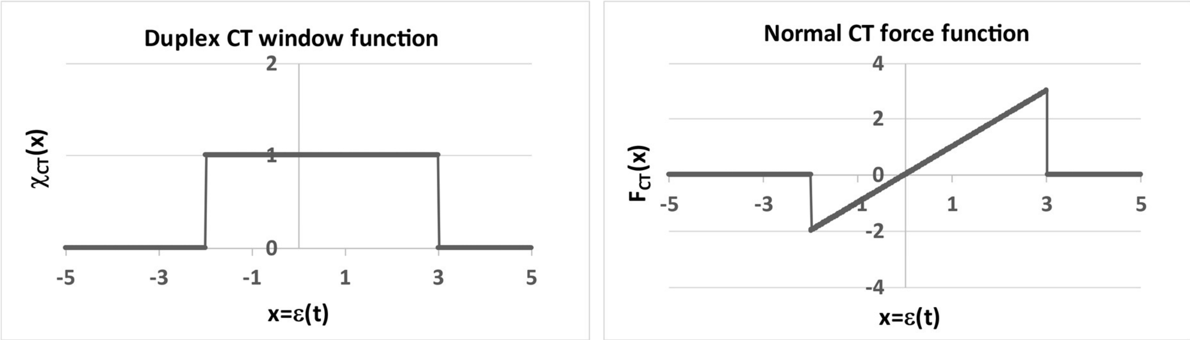
The monotonic normal (compressive ($u \leq 0$) or tensile ($0 \leq u$)) force, $F_n(u)$, arising in the n^{th} fiber subjected to monotonic normal strain load can be calculated with Eqs. (25), (11) and (12), which is proportional to the normal engineering stress, $f_n(u)$ (Figure 4):

$$F_n(u) = A_1(u)\sigma_n(u) = A_0\psi(u)\sigma_n(u) = A_0\psi(u)E_n u \chi_{CT}(u, -\varepsilon_{Cn}, \varepsilon_{Tn}) = A_0 f_n(u) \tag{27}$$

Correspondingly, applying Eqs. (26), (13) and (14), the shear force, $F_{Sn}(w)$, and the engineering shear stress, $f_{Sn}(w)$, can be defined:

$$F_{Sn}(w) = S_0\tau_n(w) = S_0 G_n w \chi_{-S,S}(u, -\varepsilon_{-Sn}, \varepsilon_{Sn}) = S_0 f_{Sn}(u) \tag{28}$$

where $f_{Sn}(w) = \tau_n(w)$ for every $w \geq 0$.



(a) (b)

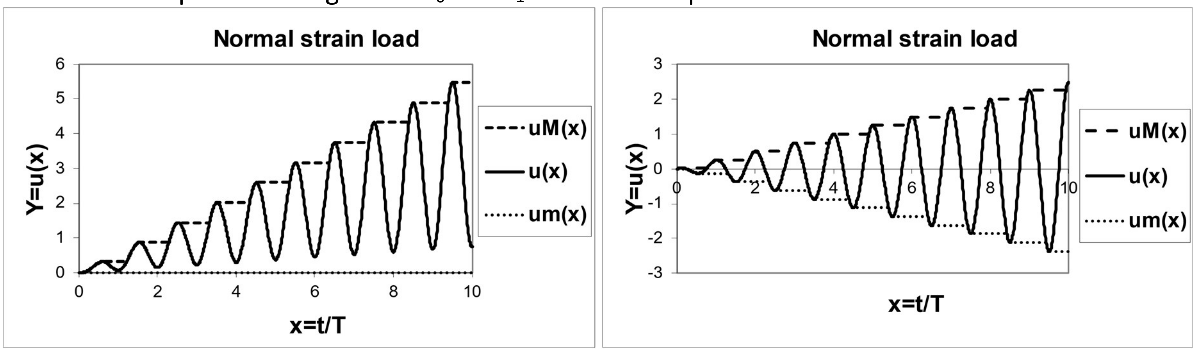
Figure 4 Example of the normal duplex WF (a) and the normalized compressive–tensile force process (b) of a single fiber as a response to the monotone increasing normal strain load ($k(x)=x$; $\varepsilon_{CB}=2$, $\varepsilon_{TB}=3$)

2.2.3 Destructive alternating load – Stochastic mechanical behavior

Previously, we used monotone strain loads. Now, let us consider a pulsating ($x\geq 0$) (Eq. (29) and Figure 5.a) or alternating ($-\infty<x<\infty$) (Eq. (30) and Figure 5.b) strain load. As an example, we use simple sinusoidal functions of linearly increasing amplitude and mean value, which is normalized by the mean breaking strain of the fibers (ε_B) (Figure 5):

$$x(t)=\frac{\varepsilon(t)}{\varepsilon_B}=\frac{1}{4}\frac{t}{T}\left(c_0+c_1 cos\left(2\pi\frac{t}{T}\right)\right) \tag{29-30}$$

where T is the period during which c_0 and c_1 are constant parameters.



(a) (b)

Figure 5 Examples of the pulsating (T=1, c_0=0.78; c_1=-0.6) (a) and the alternating (T=1, c_0=0; c_1=0.6) (b) sinusoidal function of linearly increasing amplitude and its minimum and maximum

When *pulsating or alternating* strain load is used (Figure 5), as an effect of the extreme values, it may occur that after some fiber breakage, the load decreases. However, the broken fibers do not recover, consequently reliability remains low, due to the fiber failures. To take this into account, we need to create a kind of minimum-preserving reliability function.

Definition:

The strain loads $u(t)$ or $w(t)$ are called ***alternating*** if they take up both negative and positive values in time. Minimum (index: m) and maximum (index: M) of the alternating normal and shear strain loads, $u(t)$ and $w(t)$, in the time interval $[0,t]$ are defined as follows (Figure 5):

$$u_m(t)=\min_{0\leq t'\leq t} u(t')\leq u(t)\leq u_M(t)=\max_{0\leq t'\leq t} u(t') \tag{31}$$

$$w_m(t)=\min_{0\leq t'\leq t} w(t')\leq w(t)\leq w_M(t)=\max_{0\leq t'\leq t} w(t') \tag{32}$$

The extremal functions above are monotone and for example, $u_m(t) \equiv 0$ when $u(t)$ is non-negative and pulsating (Figure 5.a).
Definition
For alternating compressive and tensile strain loads, the so-called ***memory window functions*** (MWF) of the n^{th} fiber *preserve their minimum values* taken up in the time interval $[0,t]$ and are defined as:

$$\chi^*_{Cn}\big(u(t)\big) = \min_{0\le t'\le t} \chi_C(u(t'), -\varepsilon_{Cn}) \tag{33}$$
$$\chi^*_{Tn}\big(u(t)\big) = \min_{0\le t'\le t} \chi_T(u(t'), \varepsilon_{Tn}) \tag{34}$$

For alternating negative and positive shear strain loads, they are:

$$\chi^*_{-Sn}\big(w(t)\big) = \min_{0\le t'\le t} \chi_{-S}(w(t'), -\varepsilon_{-Sn}) \tag{35}$$
$$\chi^*_{Sn}\big(w(t)\big) = \min_{0\le t'\le t} \chi_S(w(t'), \varepsilon_{Sn}) \tag{36}$$

Statement:
For alternating compressive and tensile strain loads, the MWFs of the n^{th} fiber *preserve their minimum values* taken up in time interval $[0,t]$. It can be obtained as:

$$\chi^*_{Cn}\big(u(t)\big) = \min_{0\le t'\le t} \chi_C(u(t'), -\varepsilon_{Cn}) = \chi_C(u_m(t), -\varepsilon_{Cn}) = \begin{cases} 1, & -\varepsilon_{Cn} < u_m(t) \\ 0, & u_m(t) \le -\varepsilon_{Cn} \end{cases} \tag{37}$$
$$\chi^*_{Tn}\big(u(t)\big) = \min_{0\le t'\le t} \chi_T(u(t'), \varepsilon_{Tn}) = \chi_T(u_M(t), \varepsilon_{Tn}) = \begin{cases} 1, & u_M(t) < \varepsilon_{Tn} \\ 0, & \varepsilon_{Tn} \le u_M(t) \end{cases} \tag{38}$$

For alternating negative and positive shear strain loads, they are:

$$\chi^*_{-Sn}\big(w(t)\big) = \min_{0\le t'\le t} \chi_{-S}(w(t'), -\varepsilon_{-Sn}) = \chi_{-S}(w_m(t), -\varepsilon_{-Sn}) = \begin{cases} 1, & -\varepsilon_{-Sn} < w_m(t) \\ 0, & w_m(t) \le -\varepsilon_{-Sn} \end{cases} \tag{39}$$
$$\chi^*_{Sn}\big(w(t)\big) = \min_{0\le t'\le t} \chi_S(w(t'), \varepsilon_{Sn}) = \chi_S(w_M(t), \varepsilon_{Sn}) = \begin{cases} 1, & w_M(t) < \varepsilon_{Sn} \\ 0, & \varepsilon_{Sn} \le w_M(t) \end{cases} \tag{40}$$

Proof: It is enough to see, for example Eq. (38) (or (37)). The first expression of $\chi^*_{Tn}\big(u(t)\big)$ is given by Eq. (34) (or (33)). Eq. (A2) (or (A4)) in Appendix A1 provides the second expression of Eq. (38) (or (37)) while the definition of χ_T given by Eq. (18) (or (17)) gives the third one:

$$\min_{0\le t'\le t} \chi_T(u(t'), \varepsilon_{Tn}) = \chi_T(u_M(t), \varepsilon_{Tn}) = \begin{cases} 1, & u_M(t) < \varepsilon_{Tn} \\ 0, & \varepsilon_{Tn} \le u_M(t) \end{cases}$$

Consequently, the minimum of the window function in [0,t] occurs at the maximum value of the strain load, $u_M(t)$, and this minimum equals 1 if and only if $u_M(t)$ is less than the tensile breaking strain of the fiber, which is the statement given by Eq. (38).
Eqs. (39) and (40) can be shown similarly. ☐
Definition:
For alternating normal and shear strain loads, the so-called normal and shear ***duplex memory window functions*** (duplex MWF) are defined as:

$$\chi^*_{CTn}\big(u(t)\big) = \chi^*_{CT}(u(t), -\varepsilon_{Cn}, \varepsilon_{Tn}) = \min_{0\le t'\le t} \chi_{CT}(u(t'), -\varepsilon_{Cn}, \varepsilon_{Tn}) \tag{41}$$
$$\chi^*_{-S,Sn}\big(w(t)\big) = \chi^*_{-S,S}(w(t), -\varepsilon_{-Sn}, \varepsilon_{Sn}) = \min_{0\le t'\le t} \chi_{-S,S}(w(t'), -\varepsilon_{-Sn}, \varepsilon_{Sn}) \tag{42}$$

Statement:

For compressive and tensile alternating strain loads, the normal duplex MWF function, χ^*_{CTn}, can be produced as the product of χ^*_{Cn} and χ^*_{Tn}:

$$\chi^*_{CTn}\big(u(t)\big) = \chi^*_{CT}(u(t), -\varepsilon_{Cn}, \varepsilon_{Tn}) = \chi^*_{Cn}\big(u(t)\big)\chi^*_{Tn}\big(u(t)\big) \tag{43}$$

For negative and positive alternating shear strain loads, the shear duplex MWF, $\chi^*_{-S,Sn}$, can be produced as the product of χ^*_{-Sn} and χ^*_{Sn}:

$$\chi^*_{-S,Sn}\big(w(t)\big) = \chi^*_{-S,S}(w(t), -\varepsilon_{-Sn}, \varepsilon_{Sn}) = \chi^*_{-Sn}\big(w(t)\big)\chi^*_{Sn}\big(w(t)\big) \tag{44}$$

Proof: Considering the definitions and statements in Eqs. (21)-(24), (33)-(40), (41) and (42), and utilizing Eq. (A2) in Appendix A1, the statements in Eqs. (43) and (44) can be seen like the following for normal strain load. For example, with the use of Eqs. (41) and (24), Eq. (43) can be reformulated:

$$\chi^*_{CTn}\big(u(t)\big) = \min_{0\le t'\le t} \chi_{CT}(u(t'), -\varepsilon_{Cn}, \varepsilon_{Tn}) = \min_{0\le t'\le t} \chi_C(u(t'), -\varepsilon_{Cn})\chi_T(u(t'), \varepsilon_{Tn})$$

Considering Eqs. (33) and (34), and the fact that both χ_C and χ_T take up values 0 or 1, the minimum of their product in [0,t] may be 1 or 0 similar to that in Figure 4.a. It is 1 if and only if $-\varepsilon_{Cn}<u_m(t')$ and $u_M(t')<\varepsilon_{Tn}$ or 0 if and only if $u_m(t')\le-\varepsilon_{Cn}$ or $\varepsilon_{Tn}\le u_M(t')$ at any $0\le t'\le t$. Utilizing Eqs. (21), (23), (41), and (A2) and (A4) in Appendix A1 provides the statement in Eq. (42).

$$\chi^*_{CTn}\big(u(t)\big) = \min_{0\le t'\le t} \chi_{CT}(u(t'), -\varepsilon_{Cn}, \varepsilon_{Tn}) = \min_{0\le t'\le t} \chi_C(u(t'), -\varepsilon_{Cn})\chi_T(u(t'), \varepsilon_{Tn}) =$$
$$= \begin{cases} 1, & -\varepsilon_{Cn} < u_m(t) \; and \; u_M(t) < \varepsilon_{Tn} \\ 0, & u_m(t) \le -\varepsilon_{Cn} \;\; or \;\; \varepsilon_{Tn} \le u_M(t) \end{cases} = \chi_C(u_m(t), -\varepsilon_{Cn})\chi_T(u_M(t), \varepsilon_{Tn}) = \chi^*_{Cn}\big(u(t)\big)\chi^*_{Tn}\big(u(t)\big)$$

since the equality for the window function values (1 and 0) stands for every $t\ge0$.
Eq. (43) can be shown similarly. ☐

2.3 Mechanical behavior of the fiber bundles

The force and stress properties of the fiber bundle can be obtained by adding or averaging those of the single fibers.

2.3.1 Monotone destructive load

Empirical normal and shear reliability functions for monotone destructive load

Applying Eqs. (27) and (28), the total *normal and shear forces* of the fiber bundle are given by:

$$F(u) = \textstyle\sum_{n=1}^{N} F_n(u) = NA_0\psi(u)\frac{1}{N}\sum_{n=1}^{N}\sigma_n(u) = NA_0\psi(u)\sigma(u) \tag{45}$$
$$F_S(w) = \textstyle\sum_{n=1}^{N} F_{Sn}(w) = NS_0\frac{1}{N}\sum_{n=1}^{N}\tau_n(w) = NS_0\tau(w) \tag{46}$$

Where $\sigma(u)$ is the average fiber stress $\tau(w)$, the average fiber shear stress of the fiber bundle in the fiber direction. From the above and Eqs. (27) and (28), the ***mean normal and shear stresses*** acting in the fiber bundle are:

$$\sigma(u) = \tfrac{1}{N}\textstyle\sum_{n=1}^{N}\sigma_n(u) = u\frac{1}{N}\sum_{n=1}^{N}E_n\chi_{CT}(u, -\varepsilon_{Cn}, \varepsilon_{Tn}) = u\kappa_{CT}(u) \tag{47}$$
$$\tau(w) = \tfrac{1}{N}\textstyle\sum_{n=1}^{N}\tau_n(w) = w\frac{1}{N}\sum_{n=1}^{N}G_n\chi_{-S,S}(w, -\varepsilon_{-Sn}, \varepsilon_{Sn}) = w\kappa_{-S,S}(w) \tag{48}$$

where $\kappa_{CT}(u)$, and $\kappa_{-S,S}(w)$ are the *mean normal and shear duplex modulus functions* of the bundle, respectively, which include the damage effects as well. When the moduli are constant in the bundle volume considered, that is, $E_n=E$ and $G_n=G$ (n=1,…,N), the modulus functions can be rewritten as:

$$\kappa_{CT}(u) = E\frac{1}{N}\sum_{n=1}^{N}\chi_{CT}(u, -\varepsilon_{Cn}, \varepsilon_{Tn}) = Er_{CT}(u) \quad (49)$$

$$\kappa_{-S,S}(w) = G\frac{1}{N}\sum_{n=1}^{N}\chi_{-S,S}(w, -\varepsilon_{-Sn}, \varepsilon_{Sn}) = Gr_{-S,S}(w) \quad (50)$$

where $r_{CT}(u)$ and $r_{-S,S}(u)$ are the ***empirical normal and shear duplex reliability functions*** (normal and shear duplex RFs) of the bundle, respectively:

$$0 \le r_{CT}(u) = \frac{1}{N}\sum_{n=1}^{N}\chi_{CT}(u, -\varepsilon_{Cn}, \varepsilon_{Tn}) \le 1 \quad (51)$$

$$0 \le r_{-S,S}(w) = \frac{1}{N}\sum_{n=1}^{N}\chi_{-S,S}(w, -\varepsilon_{-S}\ \ , \varepsilon_{Sn}) \le 1 \quad (52)$$

Let us introduce the empirical *compressive* (r_C) and *tensile* (r_T) (*normal*) as well as the *negative* (r_{-S}) and *positive* (r_S) *shear* (single) ***reliability functions*** (RFs):

$$0 \le r_C(u) = \frac{1}{N}\sum_{n=1}^{N}\chi_C(u, -\varepsilon_{Cn}) \le 1 \quad (53)$$

$$0 \le r_T(u) = \frac{1}{N}\sum_{n=1}^{N}\chi_T(u, \varepsilon_{Tn}) \le 1 \quad (54)$$

$$0 \le r_{-S}(w) = \frac{1}{N}\sum_{n=1}^{N}\chi_{-S}(w, -\varepsilon_{-Sn}) \le 1 \quad (55)$$

$$0 \le r_S(w) = \frac{1}{N}\sum_{n=1}^{N}\chi_S(w, \varepsilon_{Sn}) \le 1 \quad (56)$$

Statement:
The empirical normal reliability function (n. RF) can be calculated as the product of the compressive and tensile empirical reliability functions:

$$r_{CT}(u) = r_C(u)r_T(u) \quad (57)$$

Similarly, the empirical shear reliability function (s. RF) can be calculated as the product of the negative and positive empirical reliability functions:

$$r_{-S,S}(u) = r_{-S}(u)r_S(u) \quad (58)$$

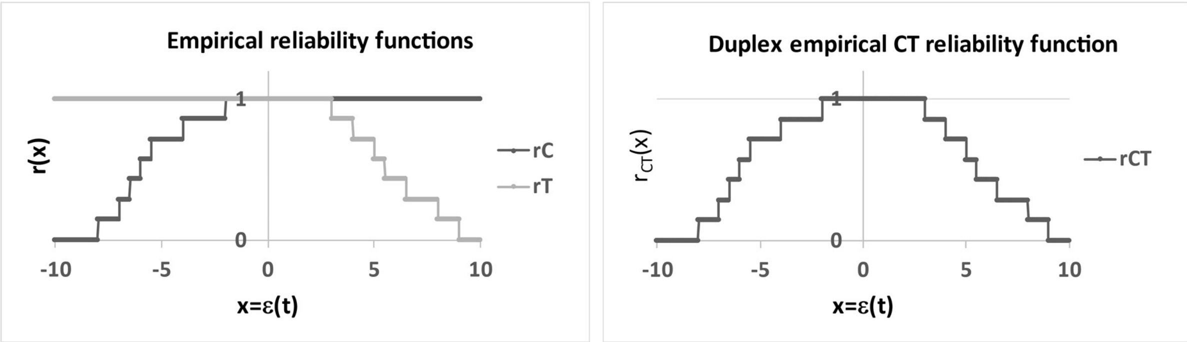
Proof: It is enough to prove Eq. (57). Taking into consideration the definition of the window functions and the fact that, because of averaging, r_T and r_S equal 1 if u<0, and r_C and r_{-S} equal 1 if u>0, let us detail the expression of r_{CT} with Eq. (57):

$$r_{CT}(u) = \begin{cases} r_C(u)\cdot 1, & u<0 \\ 1, & u=0 \\ 1\cdot r_T(u), & 0<u \end{cases} = \begin{cases} r_C(u), & u<0 \\ 1, & u=0 \\ 1, & 0<u \end{cases} \cdot \begin{cases} 1, & u<0 \\ 1, & u=0 \\ r_T(u), & 0<u \end{cases} = r_C(u)r_T(u)$$

For shear strain load, Eq. (58) can be seen in a similar way. ☐

Graphical demonstration

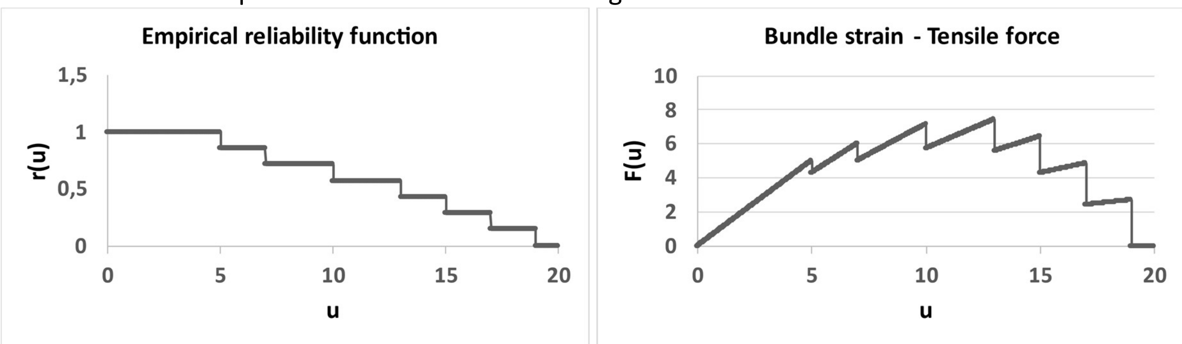
The empirical reliability functions as the mean of the fiber window functions give the decreasing fraction of the fibers intact at the strain load u. Figure 6 shows an example of them when the E-bundle consists of 7 fibers.



(a) (b)

Figure 6 Example of the empirical compressive and tensile reliability functions of a 7-fiber bundle (a) and its empirical normal reliability function (b)

As another example, Figure 7 shows the empirical tensile reliability function of a 7-fiber bundle and the tensile force response to a monotone increasing strain load.



(a) (b)

Figure 7 Example of the empirical reliability function and the tensile force response of a 7-fiber bundle ($k(x)=x$; ε_{Ti} = 5, 7, 10, 13, 15, 17, 19)

All the mechanical parameters of the fibers are stochastic variables, therefore the functions introduced and discussed above are un-stationary *stochastic processes*.

Expected value process of the normal and shear stresses for monotone destructive load

In the case of the usual monotone strain load, the expectations of the window function of the single fibers are related reliability functions of the fiber bundles, which can be given by the distribution functions of the corresponding breaking strains of the fibers. We show that through two statements.

Statement

The expectations of the related window functions of the fibers are given by the distribution functions of the compressive or tensile normal (ε_{Cn} and ε_{Tn}), or the negative and positive shear (ε_{-Sn} and ε_{Sn}) breaking strains of the fibers:

$$\mathbb{E}\big(\chi(u,-\varepsilon_{Cn})\big) = P_{-\varepsilon_C}(u) \tag{59}$$

$$\mathbb{E}\big(\chi(u,\varepsilon_{Tn})\big) = 1 - P_{\varepsilon_T}(u) \tag{60}$$

$$\mathbb{E}\big(\chi_{-S}(w,-\varepsilon_{-S})\big) = P_{-\varepsilon_{-S}}(w) \tag{61}$$

$$\mathbb{E}\big(\chi_S(w,\varepsilon)\big) = 1 - P_{\varepsilon_S}(w) \tag{62}$$

where in general, $\mathbb{E}$(X) denotes the expected value of the stochastic variable X.

Proof: It is enough to prove the normal strain load cases since those for the shear loads can be seen in a similar way. Based on the definitions of the normal window functions given by Eqs. (17) and (18), we obtain for their expectations:

$$\mathbb{E}\big(\chi(u,-\varepsilon_{Cn})\big) = 1\cdot\mathbb{P}(\chi_C = 1) + 0\cdot\mathbb{P}(\chi_C = 0) = \mathbb{P}(\chi_C = 1) = \mathbb{P}(-\varepsilon_{Cn} < u) = P_{-\varepsilon_C}(u)$$

$$\mathbb{E}(\chi(u,\varepsilon_{Tn})) = 1\cdot\mathbb{P}(\chi_T = 1) + 0\cdot\mathbb{P}(\chi_T = 0) = \mathbb{P}(\chi_T = 1) = \mathbb{P}(\varepsilon_{Tn} \geq u) = 1 - P_{\varepsilon_T}(u)$$

where $\mathbb{P}$(A) denotes the probability of event A. ☐

A similar statement is true for the expectation of the empirical reliability functions.

Statement

Let $R_C(u)$ and $R_T(u)$, and $R_{-S}(u)$ and $R_S(u)$ denote the expectations of the compressive and tensile normal, and negative and positive shear empirical reliability functions (RFs) of the fiber bundles (Eqs. (53)-(56)), respectively. They are identical with the expectations of the related window functions:

$$R_C(u) = \mathbb{E}(r_C(u)) = \mathbb{E}(\chi(u,-\varepsilon_{Cn})) = P_{-\varepsilon_C}(u) \tag{63}$$
$$R_T(u) = \mathbb{E}(r_T(u)) = \mathbb{E}(\chi(u,\varepsilon_{Tn})) = 1 - P_{\varepsilon_T}(u) \tag{64}$$
$$R_{-S}(u) = \mathbb{E}(r_{-S}(u)) = \mathbb{E}(\chi(u,-\varepsilon_{-Sn})) = P_{-\varepsilon_{-S}}(u) \tag{65}$$
$$R_S(u) = \mathbb{E}(r_S(u)) = \mathbb{E}(\chi(u,\varepsilon_{Sn})) = 1 - P_{\varepsilon_S}(u) \tag{66}$$

Proof: Now, it is enough to see one of them for example that for the normal tensile strain load. Using the definition of $r_T(u)$ given by Eq. (54) and utilizing the linearity of the expected value operator $\mathbb{E}(.)$[41, 42] – meaning that $\mathbb{E}(.)$ and Σ are exchangeable – and the assumption that the distribution function of the breaking strain is the same for the fibers, we obtain:

$$R_T(u) = \mathbb{E}(r_T(u)) = \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}(\chi(u,\varepsilon_{Tn})) = \mathbb{E}(\chi(u,\varepsilon_{Tn})) = 1 - P_{\varepsilon_T}(u)$$

where the last equation is given by Eq. (60). ☐

Similarly, the expectation of the empirical normal or shear reliability functions is the product of the expected empirical compressive and tensile normal or the negative and positive shear reliability functions, respectively, as in the next statement.

Statement

The expectation of the empirical normal and the shear duplex RFs denoted by $R_{CT}(u)$ and $R_{-S,S}(u)$ can be expressed with the following products:

$$R_{CT}(u) = \mathbb{E}(r_{CT}(u)) = \mathbb{E}(\chi_{CT}(u,-\varepsilon_{Cn},\varepsilon_{Tn})) = R_C(u)R_T(u) \tag{67}$$
$$R_{-S,S}(u) = \mathbb{E}\left(r_{-S,S}(u)\right) = \mathbb{E}\left(\chi_{-S,S}(u,-\varepsilon_{-Sn},\varepsilon_{Sn})\right) = R_{-S}(u)R_S(u) \tag{68}$$

Proof: The expected value of Eqs. (57) and (58) provides Eqs. (67) and (68) since, according to the assumptions for the breaking strains that are independent stochastic variables, the factors of $r_{CT}(u)$ or $r_{-S,S}(u)$ are also independent of each other. The internal expected values in Eqs. (67) and (68) are given by the expectations of Eqs. (51) and (52). ☐

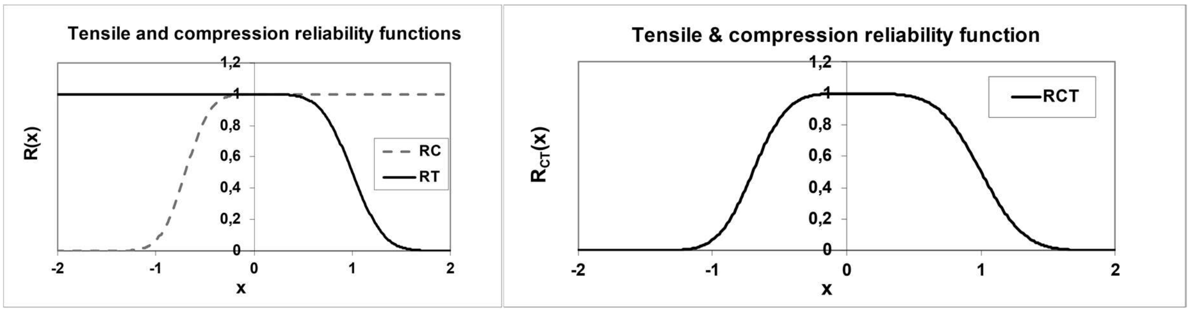
Using Eqs. (47) and) (48) and utilizing the linearity of the expected value operator $\mathbb{E}(.)$[29, 30] —meaning that $\mathbb{E}(.)$ and Σ are exchangeable—and the independency of E_n or G_n and the related breaking strains, the expected value of the total normal and shear stresses of the corresponding fiber bundles are given by:

$$\bar{\sigma}(u) = \mathbb{E}(\sigma(u)) = u\mathbb{E}(\kappa_{CT}(u)) = u\tfrac{1}{N}\textstyle\sum_{n=1}^{N}\mathbb{E}(E_n)\mathbb{E}[\chi_{CT}(u,-\varepsilon_{Cn},\varepsilon_{Tn})] = \bar{E}uR_{CT}(u) \tag{69}$$
$$\bar{\tau}(w) := \mathbb{E}(\tau(w)) = w\mathbb{E}\left(\kappa_{-S,S}(w)\right) = w\tfrac{1}{N}\textstyle\sum_{n=1}^{N}\mathbb{E}(G_n)\mathbb{E}[\chi_{-S,S}(w,-\varepsilon_{-Sn},\varepsilon_{Sn})] = \bar{G}wR_{-S,S}(w) \tag{70}$$

where $\bar{E}$ and $\bar{G}$ are the mean tensile and shear moduli of the fibers, respectively.

Graphical demonstration

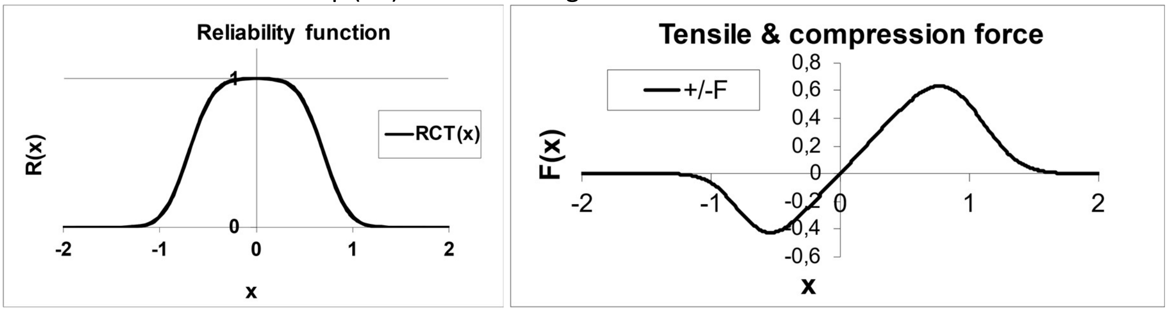
Eq. (67) is demonstrated with an example in Figure 8, where the distribution functions of the compressive and tensile breaking strains were considered normal with different expectations and standard deviations.



(a) (b)

Figure 8 Example of the expected compressive and tensile reliability functions of a fiber bundle ($\mathbb{E}(\varepsilon_C)$=2.5, $\mathbb{E}(\varepsilon_T)$=3.5) (a) and its expected double normal reliability function (b)

In Figure 9, the distribution of the compressive (ε_C) and tensile (ε_T) breaking strains were also regarded as normal distribution but with parameters N(-0,7; 0,2^2) and N(1; 0,25^2), respectively. The related reliability function of the E-bundle can be seen in Figure 9.a, while the expected compressive–tensile force curve calculated with Eq. (69) is shown in Figure 9.b.



(a) (b)

Figure 9 The duplex normal reliability function (a) and the response force process the E-bundle gives to a monotone increasing strain load (b)

2.3.2 Alternating destructive loads

Empirical normal and shear reliability functions for an alternating destructive load

For an alternating strain load, Eqs. (45) and (46) also provide the bundle force formally. However, within them, the stress functions given by Eqs. (47) and (48) valid for monotone strain load should be changed because of the alternation. With the use of Eqs. (41) and (42), the total normal and shear forces of the fiber bundle are:

$$\sigma\big(u(t)\big) = \frac{1}{N}\sum_{n=1}^{N}\sigma_n\big(u(t)\big) = u(t)\frac{1}{N}\sum_{n=1}^{N}E_n\chi_{CT}^{*}\big(u(t),-\varepsilon_{Cn},\varepsilon_{Tn}\big) = u(t)\kappa_{CT}^{*}\big(u(t)\big) \qquad (71)$$

$$\tau\big(w(t)\big) = \frac{1}{N}\sum_{n=1}^{N}\tau_n\big(w(t)\big) = w(t)\frac{1}{N}\sum_{n=1}^{N}G_n\chi_{-S,S}^{*}\big(w(t),-\varepsilon_{-Sn},\varepsilon_{Sn}\big) = w(t)\kappa_{-S,S}^{*}\big(w(t)\big) \qquad (72)$$

where $\kappa_{CT}^{*}(u)$, and $\kappa_{-S,S}^{*}(w)$ are the *mean normal and shear memory modulus functions* of the bundle, respectively, which include the damage effects as well. As above, when the moduli are considered constant in the bundle volume, that is, E_n=E and G_n=G (n=1,…,N), the modulus functions given by Eqs. (49) and (50) for a monotone strain load should be rewritten too:

$$\kappa_{CT}^{*}\big(u(t)\big) = E\frac{1}{N}\sum_{n=1}^{N}\chi_{CT}^{*}\big(u(t),-\varepsilon_{Cn},\varepsilon_{Tn}\big) = Er_{CT}^{*}\big(u(t)\big) \qquad (73)$$

$$\kappa^*_{-S,S}(w(t)) = G\frac{1}{N}\sum_{n=1}^{N}\chi^*_{-S,S}(w(t),-\varepsilon_{-Sn},\varepsilon_{Sn}) = Gr^*_{-S,S}(w(t)) \tag{74}$$

where $r^*_{CT}(u)$ and $r^*_{-S,S}(u)$ are the ***empirical normal and shear duplex MRFs*** of the bundle, respectively:

$$r^*_{CT}(u) = \frac{1}{N}\sum_{n=1}^{N}\chi^*_{CT}(u(t),-\varepsilon_{Cn},\varepsilon_{Tn}) \leq 1 \tag{75}$$
$$r^*_{-S,S}(w) = \frac{1}{N}\sum_{n=1}^{N}\chi^*_{-S,S}(w(t),-\varepsilon_{-Sn},\varepsilon_{Sn}) \leq 1 \tag{76}$$

Let us introduce the ***empirical*** *compressive* (r^*_C) and *tensile* (r^*_T) ***normal*** as well as the *negative* (r^*_{-S}) and *positive* (r^*_S) ***shear*** (single) ***MRs***:

$$r^*_C(u) = \frac{1}{N}\sum_{n=1}^{N}\chi^*_C(u(t),-\varepsilon_{Cn}) \leq 1 \tag{77}$$
$$r^*_T(u) = \frac{1}{N}\sum_{n=1}^{N}\chi^*_T(u(t),\varepsilon_{Tn}) \leq 1 \tag{78}$$

$$r^*_{-S}(w) = \frac{1}{N}\sum_{n=1}^{N}\chi^*_{-S}(w(t),-\varepsilon_{-Sn}) \leq 1 \tag{79}$$
$$r^*_S(w) = \frac{1}{N}\sum_{n=1}^{N}\chi^*_S(w(t),\varepsilon_{Sn}) \leq 1 \tag{80}$$

Expected value process of the normal and shear stresses for an alternating destructive load

Using Eqs. (71), (72), (75), (76) and utilizing the exchangeability of $\mathbb{E}(.)$ and Σ, and the independency of E_n or G_n and the normal or shear breaking strains, the expected value of the total normal and shear stresses of the related fiber bundles can be obtained as follows:

$$\bar{\sigma}(u(t)) \coloneqq \mathbb{E}\left(\sigma(u(t))\right) = u(t)\mathbb{E}\left(\kappa^*_{CT}(u(t))\right) = u(t)\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}(E_n)\mathbb{E}[\chi^*_{CT}(u(t),-\varepsilon_{Cn},\varepsilon_{Tn})] = \mathbb{E}(E_n)u(t)\mathbb{E}\left(r^*_{CT}(u(t))\right) = \bar{E}u(t)R^*_{CT}(u(t)) \tag{81}$$

$$\bar{\tau}(w(t)) \coloneqq \mathbb{E}\left(\tau(w(t))\right) = w(t)\mathbb{E}\left(\kappa^*_{-S,S}(w(t))\right) = w(t)\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}(G_n)\mathbb{E}[\chi^*_{-S,S}(w(t),-\varepsilon_{-Sn},\varepsilon_{Sn})] = \mathbb{E}(G_n)w(t)\mathbb{E}\left(r^*_{-S,S}(w(t))\right) = \bar{G}w(t)R^*_{-S,S}(w(t)) \tag{82}$$

where $R^*_{CT}(u)$ and $R^*_{-S,S}(w)$ are the ***expected normal and shear duplex MRFs*** of the related bundles, respectively, which are the expected value of the empirical normal and the shear memory reliability functions $r^*_{CT}(u)$ and $r^*_{-S,S}(w)$:

$$R^*_{CT}(u(t)) = \mathbb{E}\left(r^*_{CT}(u(t))\right) = \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}[\chi^*_{CT}(u(t),-\varepsilon_{Cn},\varepsilon_{Tn})] = \mathbb{E}[\chi^*_{CT}(u(t),-\varepsilon_{Cn},\varepsilon_{Tn})] \tag{83}$$
$$R^*_{-S,S}(w(t)) = \mathbb{E}\left(r^*_{-S,S}(w(t))\right) = \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}[\chi^*_{-S,S}(w(t),-\varepsilon_{Cn},\varepsilon_{Tn})] \tag{84}$$

Similarly to Eqs. (83) and (84), the ***expected compressive and tensile*** (normal) ***MRFs***

$$R^*_C(u(t)) = \mathbb{E}\left(r^*_C(u(t))\right) = \mathbb{E}(\chi^*_C(u(t),-\varepsilon_{Cn})) \tag{85}$$
$$R^*_T(u(t)) = \mathbb{E}\left(r^*_T(u(t))\right) = \mathbb{E}(\chi^*_T(u(t),\varepsilon_{Tn})) \tag{86}$$

and the ***expected negative and positive shear MRFs*** of the fibers can be defined:

$$R^*_{-S}(w(t)) = \mathbb{E}\left(r^*_{-S}(w(t))\right) = \mathbb{E}(\chi^*_{-S}(w(t),-\varepsilon_{-Sn})) \tag{87}$$

$$R_S^*(w(t)) = \mathbb{E}\left(r_S^*(w(t))\right) = \mathbb{E}(\chi_S^*(w(t), \varepsilon_{Sn})) \tag{88}$$

Statement

The expected ***compressive and tensile*** (normal) ***MRFs*** of the fibers can be obtained as follows:

$$R_C^*(u(t)) = \mathbb{E}(\chi_C^*(u(t), -\varepsilon_{Cn})) = P_{-\varepsilon_C}(u_m(t)) = R_C(u_m(t)) = \min_{0\le t'\le t} R_C(u(t')) \tag{89}$$
$$R_T^*(u(t)) = \mathbb{E}(\chi_T^*(u(t), \varepsilon_{Tn})) = 1 - P_{\varepsilon_T}(u_M(t)) = R_T(u_M(t)) = \min_{0\le t'\le t} R_T(u(t')) \tag{90}$$

Similarly, the expected ***negative and positive shear MRFs*** are:

$$R_{-S}^*(w(t)) = \mathbb{E}(\chi_{-S}^*(w(t), -\varepsilon_{-S}\ )) = P_{-\varepsilon_{-S}}(w_m(t)) = R_{-S}(w_m(t)) = \min_{0\le t'\le t} R_{-S}(w(t')) \tag{91}$$
$$R_S^*(w(t)) = \mathbb{E}(\chi_S^*(w(t), \varepsilon_{Sn})) = 1 - P_{\varepsilon_S}(w_M(t)) = R_S(w_M(t)) = \min_{0\le t'\le t} R_S(w(t')) \tag{92}$$

Proof: This can be shown by applying the definition of the MRFs given by Eqs. (33) and (34) and utilizing the properties of the expected value and Eqs. (A2) and (A4) in Appendix A1:

$$\begin{aligned} R_C^*(u(t)) &= \mathbb{E}(\chi_C^*(u(t), -\varepsilon_{Cn})) = 1 \cdot \mathbb{P}(\chi_C^* = 1) + 0 \cdot \mathbb{P}(\chi_C^* = 0) = \mathbb{P}(\chi_C^* = 1) = \mathbb{P}(-\varepsilon_{Cn} < u_m(t)) \\ &= P_{-\varepsilon_C}(u_m(t)) = \min_{0\le t'\le t} R_C(u(t')) \\ R_T^*(u(t)) &= \mathbb{E}(\chi_T^*(u(t), \varepsilon_{Tn})) = 1 \cdot \mathbb{P}(\chi_T^* = 1) + 0 \cdot \mathbb{P}(\chi_T^* = 0) = \mathbb{P}(\chi_T^* = 1) = \mathbb{P}(\varepsilon_{Tn} \ge u_M(t)) \\ &= 1 - P_{\varepsilon_T}(u_M(t)) = \min_{0\le t'\le t} R_T(u(t')) \end{aligned}$$

so they can be given by the distribution functions of the compressive and tensile breaking strains of the fibers, respectively. For a shear strain load, Eqs. (91) and (92) can be proved in a similar way. ☐

Statement

The expected normal duplex MRF (R_{CT}^*) is the product of the compressive (R_C^*) and tensile (R_T^*) memory reliability functions:

$$R_{CT}^*(u(t)) = R_C^*(u(t))R_T^*(u(t)) \tag{93}$$

while the expected shear duplex MRF ($R_{-S,S}^*$) is the product of the compressive (R_{-S}^*) and tensile (R_S^*) memory reliability functions:

$$R_{-S,S}^*(w(t)) = R_{-S}^*(w(t))R_S^*(w(t)) \tag{94}$$

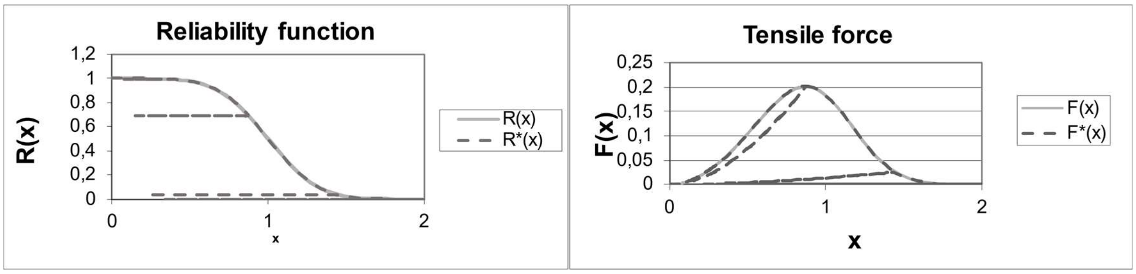
Proof: Eq. (93) can be shown with some consecutive operations. Firstly, we applied Eq. (83), then Eq. (43) to convert χ_{CT}^* into product form. Afterward, based on Eqs. (37) and (38), we utilized the facts that $u_M(t)$ and R_C are monotone increasing functions, while $u_m(t)$ and R_T are monotone decreasing (see Appendix A1) and, on the other hand, χ_C and χ_T are independent of each other:

$$\begin{aligned} R_{CT}^*(u(t)) = \mathbb{E}\left(r_{CT}^*(u(t))\right) = \mathbb{E}[\chi_{CT}^*(u(t), -\varepsilon_{Cn}, \varepsilon_{Tn})] = \mathbb{E}[\chi_C^*(u(t), -\varepsilon_{Cn})\chi_T^*(u(t), \varepsilon_{Tn})] = \\ \mathbb{E}[\chi_C(u_m(t), -\varepsilon_{Cn})\chi_C(u_M(t), \varepsilon_{Tn})] = \mathbb{E}[\chi_C(u_m(t), -\varepsilon_{Cn})]\mathbb{E}[\chi_T(u_M(t), \varepsilon_{Tn})] = \\ R_C(u_m(t))R_T(u_M(t)) = R_C^*(u(t))R_T^*(u(t)) \end{aligned}$$

For the shear strain load, a similar method can be used. ☐

Graphical demonstrations

Figure 10 shows the expected MRF (Figure 10.a) and the expected normalized tensile bundle force (Figure 10.b) as the response to the *pulsating* tensile bundle strain (see Figure 5.a), $u=x$, when the tensile characteristic function of the fibers is $k(x)=x$.

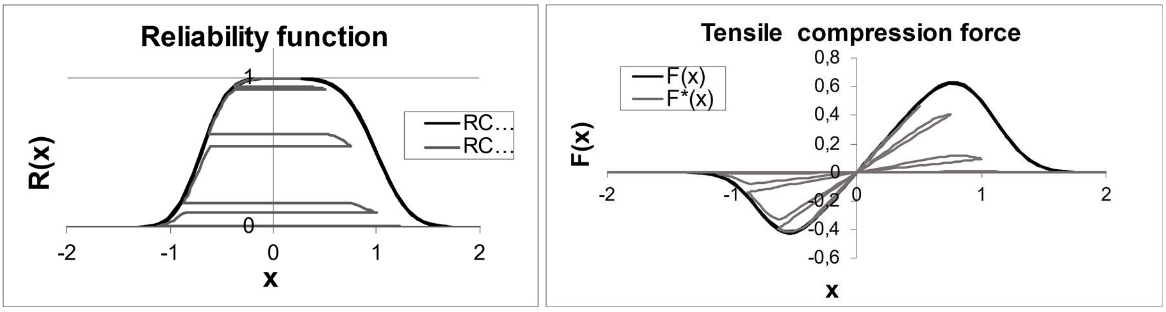


(a) (b)

Figure 10. The expected memory reliability function (a) and the expected normalized tensile bundle force (b) in the case of a pulsating tensile strain load

The more and more decreasing slope of the force process in Figure 10.b is caused by the accumulating tensile damage. After each damage sub-process, the reliability level decreases and so does the tensile stiffness of the material as well.

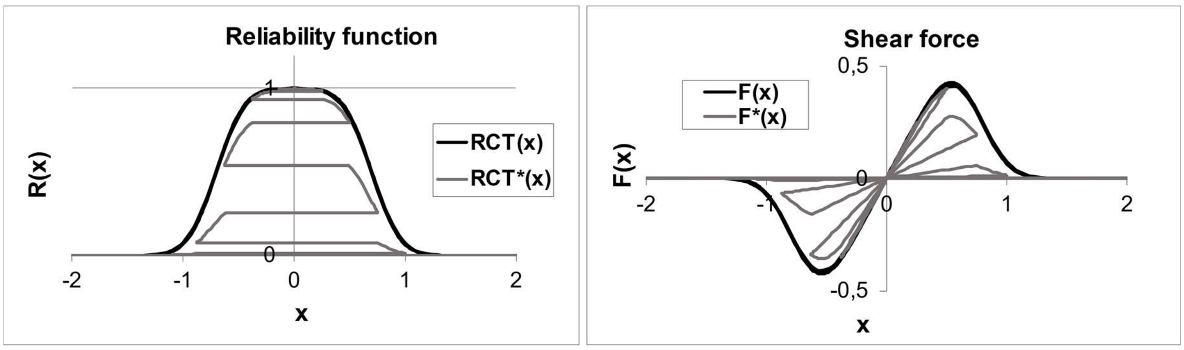
Figures 11.a and 11.b show the interaction of the failures caused by an *alternating* compressive and tensile strain load (Figure 5.b), where the compressive breaking strain is smaller than the tensile breaking strain. The steepness of the non-symmetric normalized force–deformation curve decreases more and more, which is the effect of the accumulating non-symmetric compressive and tensile damage. The loop-like shape of the stress-strain curve is the result of the interaction of the alternating tensile and compression load-caused damages reducing the global reliability.



(a) (b)

Figure 11 The expected normal MRF of the E-bundle (a) and the expected compressive–tensile force process for monotone increasing (blue lines) and alternating (red lines) normal strain load

Figures 12.a and 12.b show the interaction of the failures caused by negative and positive shear strain load, and the more and more decreasing slope of the normalized ($k_S(x)=x$) shear force–deformation function as an effect of the accumulating "quasi-symmetrical" damage.



(a) (b)

Figure 12 The expected shear MRF of the E-bundle (a) and the expected negative-positive shear force process for monotone increasing (blue lines) and alternating (red lines) shear strain load

The Cohesive Zone Method as a simplified bundle model

The simplified form of the Cohesive Zone Method (CZM)[8, 9] is the so-called bi-linear CZM model, where the traction-separation law has a triangle shape. It has often been used to describe failure phenomena in finite element (FEM) simulations[8-12]. Actually, as it is demonstrated in Figure 13, the bi-linear CZM model may be considered, which approximates the expected normalized ($k(x)=x$) tensile force of an E-bundle (F(x)) with the initial tangent (F0_(x)) of the ascending part and the inflexion tangent (F1_(x)) of the descending part.

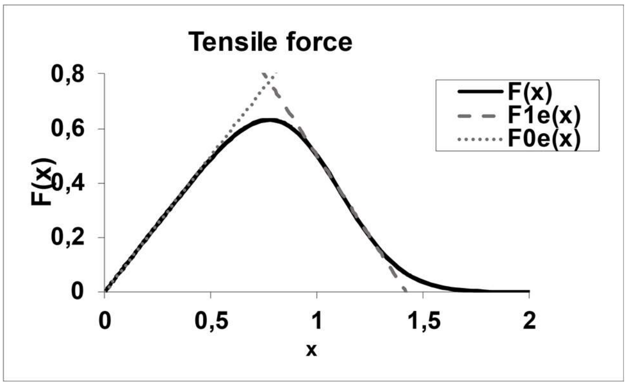


Figure 13 Demonstration of the bi-linear CZM model as the tangent approximation of the E-bundle model

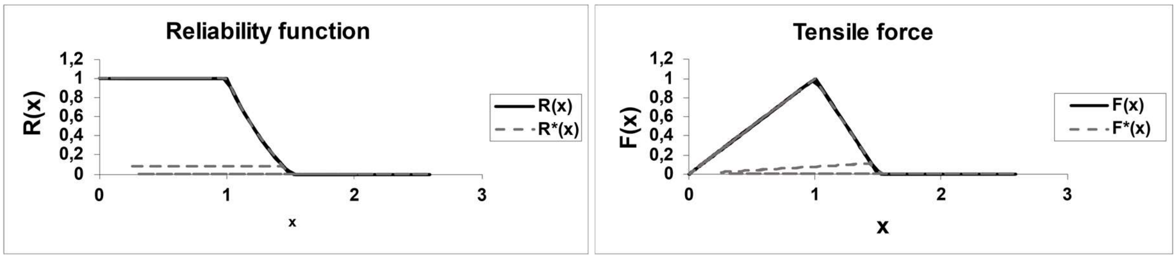
Hence, the F(x) tensile process of the CZM model can easily be given in the product form if the reliability function is as follows:

$$R(x) = \begin{cases} 1, & x = 0 \\ max\left(0; min\left(1; \frac{1}{x}\frac{x_0 - x}{x_0 - x_1}\right)\right), & x > 0 \end{cases} \qquad (95)$$

where x_0 and x_1 are constant parameters, so ($x \geq 0$):

$$F(x) = xR(x) = x \cdot max\left(0; min\left(1; \frac{1}{x}\frac{x_0 - x}{x_0 - x_1}\right)\right) = max\left(0; min\left(x; \frac{x_0 - x}{x_0 - x_1}\right)\right) \qquad (96)$$

The memory reliability function, R*(x), visible in Figure 14.a can be obtained from Eq. (29) with the use of the definition given by Eqs. (36) and (95). Figure 14.b shows the CZM normalized force–strain curve calculated with Eq. (96) and the specific force variation responding to the pulsating tensile strain load in Figure 4.a.



(a) (b)

Figure 14 The CZM reliability function (x_0=1,5; x_1=1) (a) and the related normalized tensile force as the response to a pulsating tensile strain load (b)

Corollary and remarks

•Because of the stochastic convergence of the mean[41, 42] following from the conditions for the stochastic strength parameters, it is obvious that if the number of fibers (*N*) is large enough, the empirical memory-reliability functions (r_X^* and $r_{X,Y}^*$; *X,Y*∈{*C,T,-S,S*}) tend to their expectations when *N*→∞. Consequently, for example, the following relationship approximately stands for a large enough *N*:

$$r_{CT}^*\big(u(t)\big) = \frac{1}{N}\sum_{n=1}^{N} \chi_{CT}^*\big(u(t), -\varepsilon_{Cn}, \varepsilon_{Tn}\big) \approx \mathbb{E}\left(r_{CT}^*\big(u(t)\big)\right) = R_{CT}^*\big(u(t)\big) \tag{97}$$

•On the other hand, averaging the empirical memory reliability functions obtained from different independent measurements or observations, the average stochastically converges to the expectation when the number of the measurements (*M*) tends to ∞. For example:

$$\bar{r}_{CT}^*\big(u(t)\big) = \frac{1}{M}\sum_{m=1}^{M} r_{CT,m}^*\big(u(t)\big) \approx \mathbb{E}\left(r_{CT}^*\big(u(t)\big)\right) = R_{CT}^*\big(u(t)\big) \tag{98}$$

Accordingly, the following statement can be formulated as the corollary of Eqs. (93) and (94).

Statement

The mean empirical normal MRF can be calculated approximately as the product of the mean compressive and tensile empirical reliability functions:

$$\bar{r}_{CT}^*(u) \approx \bar{r}_C^*(u)\bar{r}_T^*(u) \tag{99}$$

Similarly, the mean empirical shear MRF can be approximately calculated as the product of the mean negative and positive empirical reliability functions:

$$\bar{r}_{-S,S}^*(w) \approx \bar{r}_{-S}^*(w)\bar{r}_{-S}^*(w) \tag{100}$$

Proof: Based on the stochastic convergence of the sample mean to the finite expected value [41, 42], Eqs. (99) and (100) follow from Eqs. (93) and (94), respectively. ☐

Remarks

• Eqs. (93) and (94) are of great importance for subsequent calculations. Namely, they make it possible to determine the normal or shear MRFs using the minimum and maximum of the strain load given by Eqs. (31) and (32), which can be directly calculated from time-dependent input, that is, the controlled strain load.

$$R_{CT}^*(u(t)) = R_C^*(u(t))R_T^*(u(t)) = R_C\big(u_m(t)\big)R_T\big(u_M(t)\big) \tag{101}$$

$$R_{-S,S}^*\big(w(t)\big) = R_{-S}^*\big(w(t)\big)R_S^*\big(w(t)\big) = R_{-S}\big(w_m(t)\big)R_S\big(w_M(t)\big) \tag{102}$$

In addition, the product form of Eqs. (93) and (94) make it possible to apply useful separations for further derivations.

•It may also be mentioned that in the deterministic case the inequality (A9) in Appendix A1.3 stands instead of Eq. (93) or (94). This is because in the deterministic case, there is only one way for the minimum to decrease—to follow the graph of function *g*(*x*). In contrast to that, in the stochastic case, the reliability function is the expectation of several different failure ways including the deterministic way, as well. Hence, the graph is the upper borderline of the possible failure modes. These are demonstrated in Appendix A2.

3 LINEAR ELASTIC MATERIAL LAW INCLUDING FAILURES

To create a useful approach to the stochastic linear material law that includes the effects of the damage caused by different types of strain load, we applied E-bundle models discussed above. To make the law general, we assumed that all the controlled strain load components (ε_{kl}) may be of alternating type. To describe the stochastic damage and failure processes, we assigned an E-bundle to every strain load component. We assumed that there is no plastic strain and any type of damage can be modeled by virtual fiber breakage. Subsequently, the stochastic behavior of *matrices* or *composite* functions containing *empirical reliability functions* as elements or variables, respectively, is designated with an over-tilde~ and the memory property is denoted with an upper asterisk* (e.g. $\tilde{\mathrm{C}}^*$, $\tilde{\sigma}^*, \tilde{N}^*$). The *expectation* of these expressions has the memory property as well, therefore it is indicated by the asterisk (e.g. C^*, σ^*, N^*) (see Notation). All that enabled to apply the empirical memory reliability functions (r^*) of the related E-bundles which are special stochastic processes, and so are every matrix and every composite function that contains them as elements or a variable, respectively. Finally, we assumed that all the components of the materials such as the fibers and the matrix are linear elastic materials.

3.1 General Hooke's material model

In the case of a multiaxial controlled strain load, a linear elastic anisotropic material follows the so-called general Hooke's material model, the tensorial form of which is given by[4, 5]:

$$\mathbf{S} = \mathbf{C}:\mathbf{D} \;\leftrightarrow\; \sigma_{ij} = c_{ijkl}\varepsilon_{kl} \tag{103}$$

where D and S are the 2nd order deformation and stress tensors, respectively, while C is the 4th-order *constant* stiffness tensor and the symbol ':' denotes the so-called two-dot product[4, 5]. The right-hand side of Eq. (103) gives an equivalent description of the relationship, where σ_{ij}, c_{ijkl}, and ε_{kl} (i, j, k, l=1, 2, 3) are the scalar elements of tensors S, C, and D, respectively. Here, we applied Einstein's convention[1-5] meaning summing the product from 1 to 3 for the same factor indices. Hence, the number of the tensor elements is 3^4=9x9=81. The compacted matrix form of Eq. (103) is:

$$\underline{\sigma} = \mathrm{C}\,\underline{\varepsilon} \quad \leftrightarrow \quad \sigma_i = c_{ij}\varepsilon_j \tag{104}$$

where C is the 6x6 symmetric stiffness matrix (c_{ij}=c_{ji}) as used in composite mechanics[1-3], while, following the Voigt notation[1-5], $\underline{\sigma}$ and $\underline{\varepsilon}$ are the 6-dimensional stress and deformation vectors containing only the independent re-indexed scalar elements of S and D, respectively:

$$\underline{\sigma} = \begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{bmatrix} = \begin{bmatrix}\sigma_{11}\\\sigma_{22}\\\sigma_{33}\\\sigma_{12}\\\sigma_{23}\\\sigma_{13}\end{bmatrix} = \mathrm{C}\begin{bmatrix}\varepsilon_{11}\\\varepsilon_{22}\\\varepsilon_{33}\\\varepsilon_{12}\\\varepsilon_{23}\\\varepsilon_{13}\end{bmatrix} = \mathrm{C}\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{bmatrix} = \begin{bmatrix}c_{11}&c_{12}&c_{13}&c_{14}&c_{15}&c_{16}\\c_{21}&c_{22}&c_{23}&c_{24}&c_{25}&c_{26}\\c_{31}&c_{32}&c_{33}&c_{34}&c_{35}&c_{36}\\c_{41}&c_{42}&c_{43}&c_{44}&c_{45}&c_{46}\\c_{51}&c_{52}&c_{53}&c_{54}&c_{55}&c_{56}\\c_{61}&c_{62}&c_{63}&c_{64}&c_{65}&c_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{bmatrix} = \mathrm{C}\,\underline{\varepsilon} \tag{105}$$

3.2 FBC–based linear elastic material law

3.2.1 Description with stochastic tensors with the use of finite E-bundles

Based on the considerations made above, assume that—as a kind of phenomenological modeling—an ***empirical** duplex **MRF***, $r^*_{kl}(\varepsilon_{kl})$ belongs to each controlled alternating strain load element u_{kl}=$\varepsilon_{kl}(t)$ (k,l=1,2,3), and their product may be called ***damaging strain***:

$$\varepsilon^*_{kl} = \varepsilon_{kl}r^*_{kl}(\varepsilon_{kl}) = \varepsilon_{kl}\min_{0\le t'\le t} r_{kl}(\varepsilon_{kl}(t')) = \varepsilon_{kl}\min_{0\le t'\le t}\frac{1}{N_{kl}}\sum_{l=1}^{N_{kl}}\chi(\varepsilon_{kl}(t');\varepsilon_{-Bkl},\varepsilon_{+Bkl}) \tag{106}$$

where N_{kl} is the fiber number of the bundle assigned to the strain load ε_{kl} as well as ε_{-Bkl} and ε_{+Bkl} are the negative and positive breaking strain, which are stochastic variables. If $k=l$ then ε_{kl} is normal compressive (-B=C) or tensile (B=T) strain, while $k \neq l$ means negative or positive shear (B=S) strain.
Supposing that the scalar elements, c_{ijkl}, of tensor $\mathbb{C}$ are ***constant*** (elasticity constants), and taking into account Eqs. (103) and (106), the tensorial form of the ***material law including the damages*** is as follows:

$$\tilde{\mathbf{S}}^* = \mathbf{C}:\tilde{\mathbf{D}}^* = \mathbf{C}:\tilde{\mathbf{R}}^*:\mathbf{D} \leftrightarrow \sigma^*_{kl} = c_{ijkl}\varepsilon^*_{kl} = c_{ij\acute{k}\acute{l}}r^*_{\acute{k}\acute{l}}(\varepsilon_{kl})\varepsilon_{kl} \tag{107}$$

where $\tilde{\mathbf{R}}^*$ is the 4[th]-order duplex ***empirical*** *duplex* ***memory reliability*** (empirical DMR) ***tensor***, shortly *empirical* ***DMR tensor*** or empirical ***reliability tensor.*** The latter depends on the strain tensor $\mathbb{D}$, yet the operations of linear algebra may be applied to the product form of Eq. (107). Hence, similar to Eq. (104), the matrix form of Eq. (107) is:

$$\underline{\tilde{\sigma}}^* = \mathrm{C}\,\underline{\tilde{\varepsilon}}^* = \mathrm{C}\,\tilde{\mathrm{R}}^*\,\underline{\varepsilon} \leftrightarrow \sigma^*_i = c_{ij}\varepsilon^*_j = c_{ij}r^*_{jj}(\varepsilon_j)\varepsilon_j \tag{108}$$

where $\tilde{\mathrm{R}}^*$ is a 6x6 diagonal matrix, while $\underline{\tilde{\sigma}}^*$ and $\underline{\tilde{\varepsilon}}^*$ are 6-dimensional vectors and $r^*_j = r^*_{jj}$, as Eq. (109) shows:

$$\underline{\tilde{\sigma}}^* = \begin{bmatrix}\tilde{\sigma}^*_1\\ \tilde{\sigma}^*_2\\ \tilde{\sigma}^*_3\\ \tilde{\sigma}^*_4\\ \tilde{\sigma}^*_5\\ \tilde{\sigma}^*_6\end{bmatrix} = \begin{bmatrix}\tilde{\sigma}^*_{11}\\ \tilde{\sigma}^*_{22}\\ \tilde{\sigma}^*_{33}\\ \tilde{\sigma}^*_{12}\\ \tilde{\sigma}^*_{23}\\ \tilde{\sigma}^*_{13}\end{bmatrix} = \mathrm{C}\,\underline{\tilde{\varepsilon}}^* = \mathrm{C}\begin{bmatrix}\tilde{\varepsilon}^*_{11}\\ \tilde{\varepsilon}^*_{22}\\ \tilde{\varepsilon}^*_{33}\\ \tilde{\varepsilon}^*_{12}\\ \tilde{\varepsilon}^*_{23}\\ \tilde{\varepsilon}^*_{13}\end{bmatrix} = \mathrm{C}\begin{bmatrix}\tilde{\varepsilon}^*_1\\ \tilde{\varepsilon}^*_2\\ \tilde{\varepsilon}^*_3\\ \tilde{\varepsilon}^*_4\\ \tilde{\varepsilon}^*_5\\ \tilde{\varepsilon}^*_6\end{bmatrix} = \mathrm{C}\begin{bmatrix}r^*_1\varepsilon_1\\ r^*_2\varepsilon_2\\ r^*_3\varepsilon_3\\ r^*_4\varepsilon_4\\ r^*_5\varepsilon_5\\ r^*_6\varepsilon_6\end{bmatrix} = \mathrm{C}\begin{bmatrix}r^*_1 & 0 & 0 & 0 & 0 & 0\\ 0 & r^*_2 & 0 & 0 & 0 & 0\\ 0 & 0 & r^*_3 & 0 & 0 & 0\\ 0 & 0 & 0 & r^*_4 & 0 & 0\\ 0 & 0 & 0 & 0 & r^*_5 & 0\\ 0 & 0 & 0 & 0 & 0 & r^*_6\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{bmatrix} = \mathrm{C}\,\tilde{\mathrm{R}}^*\underline{\varepsilon} \tag{109}$$

3.2.2 Expectation and variance tensors

Expectation

The expected value of the stress tensor $\tilde{\mathbf{S}}^*$ can be calculated from Eq. (107):

$$\mathbf{S}^* = \mathbb{E}[\tilde{\mathbf{S}}^*] = \mathbf{C}:\mathbb{E}[\tilde{\mathbf{D}}^*] = \mathbf{C}:\mathbb{E}[\tilde{\mathbf{R}}^*]:\mathbf{D} = \mathbf{C}:\mathbf{R}^*:\mathbf{D} \leftrightarrow \mathbb{E}[\sigma^*_{kl}] = c_{ijkl}\mathbb{E}[\varepsilon^*_{kl}] = c_{ij\acute{k}\acute{l}}R^*_{\acute{k}\acute{l}}\varepsilon_{kl} \tag{110}$$

where **R*** and R* are the *expected* ***DMR tensor*** and matrix, respectively:

$$\mathbf{R}^* = \mathbb{E}[\tilde{\mathbf{R}}^*], \qquad \mathrm{R}^* = \left[R^*_{\acute{k}\acute{l}}\right]^3_{\acute{k},\acute{l}=1} = \left[\mathbb{E}\left[r^*_{\acute{k}\acute{l}}(\varepsilon_{\acute{k}\acute{l}})\right]\right]^3_{\acute{k},\acute{l}=1} \tag{111}$$

The matrix form of Eq. (112) is similar to Eq. (109):

$$\underline{\sigma}^* = \begin{bmatrix}\sigma^*_1\\ \sigma^*_2\\ \sigma^*_3\\ \sigma^*_4\\ \sigma^*_5\\ \sigma^*_6\end{bmatrix} = \begin{bmatrix}\sigma^*_{11}\\ \sigma^*_{22}\\ \sigma^*_{33}\\ \sigma^*_{12}\\ \sigma^*_{23}\\ \sigma^*_{13}\end{bmatrix} = \mathrm{C}\,\underline{\varepsilon}^* = \mathrm{C}\begin{bmatrix}\varepsilon^*_{11}\\ \varepsilon^*_{22}\\ \varepsilon^*_{33}\\ \varepsilon^*_{12}\\ \varepsilon^*_{23}\\ \varepsilon^*_{13}\end{bmatrix} = \mathrm{C}\begin{bmatrix}\varepsilon^*_1\\ \varepsilon^*_2\\ \varepsilon^*_3\\ \varepsilon^*_4\\ \varepsilon^*_5\\ \varepsilon^*_6\end{bmatrix} = \mathrm{C}\begin{bmatrix}R^*_1\varepsilon_1\\ R^*_2\varepsilon_2\\ R^*_3\varepsilon_3\\ R^*_4\varepsilon_4\\ R^*_5\varepsilon_5\\ R^*_6\varepsilon_6\end{bmatrix} = \mathrm{C}\begin{bmatrix}R^*_1 & 0 & 0 & 0 & 0 & 0\\ 0 & R^*_2 & 0 & 0 & 0 & 0\\ 0 & 0 & R^*_3 & 0 & 0 & 0\\ 0 & 0 & 0 & R^*_4 & 0 & 0\\ 0 & 0 & 0 & 0 & R^*_5 & 0\\ 0 & 0 & 0 & 0 & 0 & R^*_6\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{bmatrix} = \mathrm{C}\,\mathrm{R}^*\underline{\varepsilon} \tag{112}$$

It should be noted that matrix C is symmetric[1-3], but, in general, matrix CR* is not. As for the reliability functions R_i^* (i=1,…,6), they can be estimated from mechanical tests applying suitable strain loads. The determinant of matrix R* can be defined as a kind of ***global reliability*** for the material:

$$0 \leq R_{Glob}^*(t) = \ det\mathrm{R}^* = \prod_{i=1}^{6} R_i^*(\varepsilon_i(t)) \leq \min_{1\leq i\leq 6} R_i^*(\varepsilon_i(t)) \leq 1 \tag{113}$$

Global reliability is less than the minimum of the component reliability functions R_i^* (i=1,…,6) or equal if the others are 1 therefore it can be used as an exact ***damage limit*** for the material. For example, the material can be considered damaged or failed at a strain load level in time interval [0,t] if global reliability is not greater than a given critical value, R_{crit}.

Variance

The stress tensor can be centralized with its expected value:

$$\tilde{\mathbf{S}}^* - \mathbf{S}^* = \mathbf{C}:\tilde{\mathbf{R}}^*:\mathbf{D} - \mathbf{C}:\mathbb{E}\left[\tilde{\mathbf{R}}^*\right]:\mathbf{D} = \mathbf{C}:\left(\tilde{\mathbf{R}}^* - \mathbf{R}^*\right):\mathbf{D} \tag{114}$$

the elements of which are:

$$\sigma_{kl}^* - \mathbb{E}[\sigma_{kl}^*] = c_{ijkl}(\varepsilon_{kl}^* - \mathbb{E}[\varepsilon_{kl}^*]) = c_{ij\acute{k}\acute{l}}\left(r_{\acute{k}\acute{l}}^* - \mathbb{E}\left[r_{\acute{k}\acute{l}}^*\right]\right)\varepsilon_{kl} = c_{ij\acute{k}\acute{l}}\left(r_{\acute{k}\acute{l}}^* - R_{\acute{k}\acute{l}}^*\right)\varepsilon_{kl} \tag{115}$$

Otherwise, the matrix form of the centralized stress tensor is:

$$\underline{\tilde{\sigma}}^* - \underline{\sigma}^* = \mathrm{C}\left(\tilde{\mathrm{R}}^* - \mathrm{R}^*\right)\underline{\varepsilon} \tag{116}$$

The variance of the 2nd order stress tensor is represented by the ***4th order covariance tensor***, which is the expectation of the tensorial product[4, 5] of the centralized stress tensors[42]:

$$\mathbf{D}_\sigma^{*2} = \mathbb{D}^2\left(\tilde{\mathbf{S}}^*\right) = \mathbb{E}\left[\left(\tilde{\mathbf{S}}^* - \mathbf{S}^*\right)\otimes\left(\tilde{\mathbf{S}}^* - \mathbf{S}^*\right)\right] = \mathbb{E}\left[\tilde{\mathbf{S}}^*\otimes\tilde{\mathbf{S}}^* - \mathbf{S}^*\otimes\tilde{\mathbf{S}}^* - \tilde{\mathbf{S}}^*\otimes\mathbf{S}^* + \mathbf{S}^*\otimes\mathbf{S}^*\right] = \\ = \mathbb{E}\left[\tilde{\mathbf{S}}^*\otimes\tilde{\mathbf{S}}^*\right] - \mathbf{S}^*\otimes\mathbf{S}^* \tag{117}$$

where $\mathbb{D}^2$ is the variance operator and the symbol '⊗' denotes the tensorial product.
Applying Eqs. (111) and (114) to rewriting Eq. (117) yields:

$$\mathbf{D}_\sigma^{*2} = \mathbb{E}\left[\left(\mathbf{C}:\left(\tilde{\mathbf{R}}^* - \mathbf{R}^*\right):\mathbf{D}\right)\otimes\left(\mathbf{C}:\left(\tilde{\mathbf{R}}^* - \mathbf{R}^*\right):\mathbf{D}\right)\right] = \mathbb{E}\left[\left(\mathbf{C}:\tilde{\mathbf{R}}^*:\mathbf{D}\right)\otimes\left(\mathbf{C}:\tilde{\mathbf{R}}^*:\mathbf{D}\right)\right] - \\ \left(\mathbf{C}:\mathbf{R}^*:\mathbf{D}\right)\otimes\left(\mathbf{C}:\mathbf{R}^*:\mathbf{D}\right) \tag{118}$$

Rewriting Eqs. (117) and (118) provides the variance of the stress vector, that is, its covariance matrix:

$$\mathrm{D}_\sigma^{*2} = \mathbb{D}^2\left(\underline{\tilde{\sigma}}^*\right) = \mathbb{E}\left[\left(\underline{\tilde{\sigma}}^* - \underline{\sigma}^*\right)\left(\underline{\tilde{\sigma}}^* - \underline{\sigma}^*\right)^T\right] = \mathbb{E}\left[\underline{\tilde{\sigma}}^*\underline{\tilde{\sigma}}^{*T}\right] - \underline{\sigma}^*\underline{\sigma}^{*T} \tag{119}$$

or its detailed form:

$$\mathrm{D}_\sigma^{*2} = \mathbb{E}\left[\left(\mathrm{C}\left(\tilde{\mathrm{R}}^* - \mathrm{R}^*\right)\underline{\varepsilon}\right)\left(\mathrm{C}\left(\tilde{\mathrm{R}}^* - \mathrm{R}^*\right)\underline{\varepsilon}\right)^T\right] = \mathbb{E}\left[\mathrm{C}\left(\tilde{\mathrm{R}}^* - \mathrm{R}^*\right)\underline{\varepsilon}\,\underline{\varepsilon}^T\left(\tilde{\mathrm{R}}^* - \mathrm{R}^*\right)^T\mathrm{C}^T\right] = \\ = \mathrm{C}\,\mathbb{E}\left[\left(\tilde{\mathrm{R}}^* - \mathrm{R}^*\right)\underline{\varepsilon}\,\underline{\varepsilon}^T\left(\tilde{\mathrm{R}}^* - \mathrm{R}^*\right)^T\right]\mathrm{C}^T = \mathrm{C}\,\mathbb{E}\left[\tilde{\mathrm{R}}^*\underline{\varepsilon}\,\underline{\varepsilon}^T\tilde{\mathrm{R}}^{*T}\right]\mathrm{C}^T - \mathrm{C}\,\mathrm{R}^*\underline{\varepsilon}\,\underline{\varepsilon}^T\mathrm{R}^{*T}\mathrm{C}^T \tag{120}$$

Taking into account that matrix C and dyad $\underline{\varepsilon}\,\underline{\varepsilon}^T$ are symmetric matrices and matrix $\tilde{\mathrm{R}}^* - \mathrm{R}^*$ is diagonal, the elements of which are independent stochastic variables with zero expected values, therefore the expected matrix in Eq. (120) is diagonal as well:

$$\mathbb{E}\left(\left(\widetilde{\mathrm{R}}^{*}-\mathrm{R}^{*}\right)\underline{\varepsilon}\,\underline{\varepsilon}^{T}\left(\widetilde{\mathrm{R}}^{*}-\mathrm{R}^{*}\right)^{T}\right)=\left[\mathbb{E}\left((r_i^{*}-R_i^{*})(r_j^{*}-R_j^{*})\right)\varepsilon_i\varepsilon_j\right]=\left[\lambda_{ij}^{*2}\right]=\Lambda^{*2} \quad (121)$$

where the elements of matrix Λ^{*2} defined by Eq. (121) are

$$\lambda_{ij}^{*2}=\begin{cases}\mathbb{E}[(r_i^{*}-R_i^{*})^2]\varepsilon_i^2=\mathbb{D}^2(r_i^{*})\varepsilon_i^2,\ i=j\\ \mathbb{E}[(r_i^{*}-R_i^{*})]E[r_j^{*}-R_j^{*}]\varepsilon_i\varepsilon_j=0,\ i\neq j\end{cases} \quad (122)$$

Consequently, the diagonal matrix Λ^{*2} can be regarded as the product of two matrices. In the main diagonal of one of the matrices, variances $\mathbb{D}^2(r_i^{*})$ can be found while the other matrix contains squared deformations ε_i^2. Correspondingly, the simpler shape of Eq. (120) is this (C^T=C):

$$\mathrm{D}_{\sigma}^{*2}=\mathrm{C}\,\Lambda^{*2}\mathrm{C}=\mathrm{C}\,\mathrm{D}_R^{*2}diag\left(\underline{\varepsilon}\,\underline{\varepsilon}^{T}\right)\mathrm{C}=\mathrm{C}\,\mathrm{D}_R^{*2}\Lambda^2\mathrm{C} \quad (123)$$

where $\Lambda^2=diag\left(\underline{\varepsilon}\,\underline{\varepsilon}^{T}\right)$ is the diagonal part of the dyad $\underline{\varepsilon}\,\underline{\varepsilon}^{T}$ [4, 5, 42] and

$$\Lambda^{*2}=\mathrm{D}_{\sigma}^{*2}diag\left(\underline{\varepsilon}\,\underline{\varepsilon}^{T}\right)=\begin{bmatrix}\mathbb{D}^2(r_1^{*}) & 0 & 0 & 0 & 0 & 0\\ 0 & \mathbb{D}^2(r_2^{*}) & 0 & 0 & 0 & 0\\ 0 & 0 & \mathbb{D}^2(r_3^{*}) & 0 & 0 & 0\\ 0 & 0 & 0 & \mathbb{D}^2(r_4^{*}) & 0 & 0\\ 0 & 0 & 0 & 0 & \mathbb{D}^2(r_5^{*}) & 0\\ 0 & 0 & 0 & 0 & 0 & \mathbb{D}^2(r_6^{*})\end{bmatrix}\begin{bmatrix}\varepsilon_1^2 & 0 & 0 & 0 & 0 & 0\\ 0 & \varepsilon_2^2 & 0 & 0 & 0 & 0\\ 0 & 0 & \varepsilon_3^2 & 0 & 0 & 0\\ 0 & 0 & 0 & \varepsilon_4^2 & 0 & 0\\ 0 & 0 & 0 & 0 & \varepsilon_5^2 & 0\\ 0 & 0 & 0 & 0 & 0 & \varepsilon_6^2\end{bmatrix}=\mathrm{D}_{\sigma}^{*2}\Lambda^2 \quad (124)$$

Otherwise, the standard deviation matrix D_{σ}^{*} and the diagonal deformation matrix Λ are as follows:

$$\mathrm{D}_{\sigma}^{*}=\begin{bmatrix}\mathbb{D}(r_1^{*}) & 0 & 0 & 0 & 0 & 0\\ 0 & \mathbb{D}(r_2^{*}) & 0 & 0 & 0 & 0\\ 0 & 0 & \mathbb{D}(r_3^{*}) & 0 & 0 & 0\\ 0 & 0 & 0 & \mathbb{D}(r_4^{*}) & 0 & 0\\ 0 & 0 & 0 & 0 & \mathbb{D}(r_5^{*}) & 0\\ 0 & 0 & 0 & 0 & 0 & \mathbb{D}(r_6^{*})\end{bmatrix},\quad \Lambda=\begin{bmatrix}|\varepsilon_1| & 0 & 0 & 0 & 0 & 0\\ 0 & |\varepsilon_2| & 0 & 0 & 0 & 0\\ 0 & 0 & |\varepsilon_3| & 0 & 0 & 0\\ 0 & 0 & 0 & |\varepsilon_4| & 0 & 0\\ 0 & 0 & 0 & 0 & |\varepsilon_5| & 0\\ 0 & 0 & 0 & 0 & 0 & |\varepsilon_6|\end{bmatrix} \quad (125)$$

Using the latter matrices, Eq. (123) can be rewritten:

$$\mathrm{D}_{\sigma}^{*2}=\mathrm{C}\Lambda^{*2}\mathrm{C}=\mathrm{C}\,\Lambda\,\mathrm{D}_R^{*2}\Lambda\,\mathrm{C}=\mathrm{C}\,\Lambda\,\mathrm{D}_R^{*}(\mathrm{C}\,\Lambda\,\mathrm{D}_R^{*})^{T} \quad (126)$$

Based on Eqs. (123) and (126), it can be stated that the matrix $\mathrm{C}\,\Lambda\,\mathrm{D}_R^{*}$ is symmetric, then

$$\mathrm{D}_{\sigma}^{*}=(\mathrm{C}\,\Lambda^{*2}\mathrm{C})^{1/2}=\mathrm{C}\,\Lambda\,\mathrm{D}_R^{*} \quad (127)$$

or if the matrix C Λ is orthogonal then the elements of D_{σ}^{*2} are the eigen-functions[42] of D_{σ}^{*2}.
Otherwise, using Eq. (A5) in Appendix A2, matrices D_R^{*2} and D_R^{*} can be expressed with the elements of matrix R*:

$$\mathrm{D}_R^{*2} = \frac{1}{N}\begin{bmatrix} R_1^*(1-R_1^*) & 0 & 0 & 0 & 0 & 0 \\ 0 & R_2^*(1-R_2^*) & 0 & 0 & 0 & 0 \\ 0 & 0 & R_3^*(1-R_3^*) & 0 & 0 & 0 \\ 0 & 0 & 0 & R_4^*(1-R_4^*) & 0 & 0 \\ 0 & 0 & 0 & 0 & R_5^*(1-R_5^*) & 0 \\ 0 & 0 & 0 & 0 & 0 & R_6^*(1-R_6^*) \end{bmatrix}, \quad \mathrm{D}_R^* = \frac{1}{\sqrt{N}}\begin{bmatrix} \sqrt{R_1^*(1-R_1^*)} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{R_2^*(1-R_2^*)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{R_3^*(1-R_3^*)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{R_4^*(1-R_4^*)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{R_5^*(1-R_5^*)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{R_6^*(1-R_6^*)} \end{bmatrix} \tag{128}$$

3.2.3 Confidence range of the stress process

If the covariance matrix is known, the confidence range ($\mathcal{S}$) for the mean stress vector ($\underline{\overline{\sigma}}^*$) can be constructed at a given probability level (p). It depends on the controlled strain load vector ($\underline{\varepsilon}$)[41, 42]:

$$\mathbb{P}\left(\underline{\overline{\sigma}}^* \in \mathcal{S}\left(\underline{\varepsilon}\right)\right) = p \tag{129}$$

where

$$\underline{\overline{\sigma}}^* = \frac{1}{M}\sum_{m=1}^{M} \underline{\sigma}_m^* = \frac{1}{M}\sum_{m=1}^{M}\begin{bmatrix} \sigma_{1,m}^* \\ \sigma_{2,m}^* \\ \sigma_{3,m}^* \\ \sigma_{4,m}^* \\ \sigma_{5,m}^* \\ \sigma_{6,m}^* \end{bmatrix} = \begin{bmatrix} \overline{\sigma}_1^* \\ \overline{\sigma}_2^* \\ \overline{\sigma}_3^* \\ \overline{\sigma}_4^* \\ \overline{\sigma}_5^* \\ \overline{\sigma}_6^* \end{bmatrix} \tag{130}$$

and M is the number of the measurements or observations.
The fact that the covariance matrix D_R^{*2} given by Eq. (128) is diagonal means that the components of the stress vector are at least uncorrelated. Supposing that they are also independent, for example they can be considered of normal distribution, construction may be performed component by component:

$$\mathbb{P}\left(\underline{\overline{\sigma}}^* \in \mathcal{S}\left(\underline{\varepsilon}\right)\right) = \prod_{k=1}^{6} \mathbb{P}\left(\overline{\sigma}_k^* \in \mathcal{S}_k\left(\underline{\varepsilon}\right)\right) = \prod_{k=1}^{6} p_k = p \tag{131}$$

where $\mathcal{S}_k\left(\underline{\varepsilon}\right)$ is the confidence interval[42] for the stress component $\overline{\sigma}_k^*$ (k=1,…,6) at probability level p_k:

$$\mathcal{S}_k\left(\underline{\varepsilon}\right) = \left(\mathbb{E}(\sigma_k^*) - \frac{z_{M,p_k}}{\sqrt{M}}\mathbb{D}(\sigma_k^*),\ \mathbb{E}(\sigma_k^*) + \frac{z_{M,p_k}}{\sqrt{M}}\mathbb{D}(\sigma_k^*)\right) \tag{132}$$

Here, z_{M,p_k} is the critical value, and the probability p_k relates to the mean stress component $\overline{\sigma}_k^*$:

$$p_k = \mathbb{P}\left(\overline{\sigma}_k^* \in \mathcal{S}_k\left(\underline{\varepsilon}\right)\right) \tag{133}$$

The standard deviation of components σ_k^* can be obtained from matrix D_R^* in Eq. (128):

$$\mathbb{D}(\sigma_k^*) = \frac{\sqrt{R_k^*(1-R_k^*)}}{\sqrt{N}} \tag{134}$$

In addition, the confidence range $\mathcal{S}\left(\underline{\varepsilon}\right)$ is the direct product of the confidence intervals $\mathcal{S}_k\left(\underline{\varepsilon}\right)$:

$$\mathcal{S}\left(\underline{\varepsilon}\right) = \mathcal{S}_1\left(\underline{\varepsilon}\right) \times \mathcal{S}_2\left(\underline{\varepsilon}\right) \times \ldots \times \mathcal{S}_6\left(\underline{\varepsilon}\right) \tag{135}$$

It is a reasonable choice if each p_k is considered the same, that is, (k=1,…,6):

$$p_k = p^{1/6} \tag{136}$$

4 SUMMARY AND CONCLUSIONS

4.1 Summary

Modelling strain, stress and destruction of simple E-bundles as stochastic process is the basis of the description of the behavior composite materials.
The reliability and failure tests show that fibers damaged during stress do not take part in further load bearing. It is especially critical under compressive and tensile stress since damage during stress in one direction affects behavior in the other direction.
We introduced the minimum-preserving memory window functions (MWFs) defined for variate fiber parameters. The defined empirical and expected reliability functions characterize the stochastic destruction process of fibers. These functions remember the past and retain the effect of damage and failures on the stress–strain relationships when the controlled normal or shear strain load is pulsating or may decrease.
We showed that the damaging effect of the controlled alternating strain load could be taken into account with its minimum and maximum envelopes.
We proved that the duplex MWFs and the expected duplex memory reliability functions (MRFs) can be obtained as the product of those related to the compression and the tensile functions for normal strain load and the product of those related to the negative and positive shear functions for shear strain load.
Based on the properties of the MRFs and their product formula and corresponding to the conditions formulated at the very beginning of Chapter 3, we derived a stochastic material Hooke law for anisotropic linear elastic mechanical behavior with the multidimensional strain excited damage stress tensor. The model takes into account the effects of the damage and failure processes when a controlled strain load is used.
At a given probability level, a strain-dependent confidence interval can be defined for the stress components with our model.

4.2 Conclusions and remarks

-Modeling failure effects with FBC-based material law provides a special stress–strain relationship that is positive for all the finite deformations, hence there is no ultimate breakage or fracture. Consequently, for example, implicit FE simulations do not need re-meshing, just the use the FBC-based formulas.
-All that can be further developed and extended for

- controlled stress load,
- when elasticity constants (E, G, ν) are not constant but stochastic variables – for both controlled strain and stress load,
- large deformations – nonlinearity in the fiber deformations – using nonlinear E-bundles,
- linear viscoelastic behavior – using viscoelastic E-bundles.

Based on the results, a stochastic Composite Laminate Theory (CLT) can be developed that we will want to present in the second part of this paper.

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APPENDIX

A1 Theorem of the minimum-preserving functions

A1.1 Monotone decreasing function

•Let $g(x)$, ($0 \le x < \infty$), be a monotone decreasing real function, and $0 \le u(t)$, ($0 \le t < \infty$), a continuous arbitrary non-negative real function. Consider the maximum, u_M, of $u(t')$ in the interval $0 \le t' \le t < \infty$ as follows:

$$0 \le u(t) \le u_M(t) = \max_{0 \le t' \le t} u(t') \tag{A1}$$

It is obvious that $u_M(t)$ ($0 \le t$) is a monotone increasing function.
Statement:

$$g(u_M(t)) = g\left(\max_{0 \le t' \le t} u(t')\right) = \min_{0 \le t' \le t} g(u(t')) \tag{A2}$$

Proof: If $0 \le t' < t$, the next inequalities stand:

$$u_M(t') \le u_M(t) \Longrightarrow g(u_M(t')) \ge g(u_M(t))$$

because $u_M(t)$ is monotone increasing and $g(x)$ is monotone decreasing. The left-hand inequality above stands for every $t' \in [0,t]$, consequently $g(u_M(t))$ equals the minimum of $g(u(t'))$, $t' \in [0,t]$. ☐

A1.2 Monotone increasing function

•A similar statement is true when $h(x)$ is a monotone increasing function, and $u(t) \le 0$, ($0 \le t < \infty$), is a continuous arbitrary non-positive real function. Consider the minimum, u_m, of $u(t')$ in the interval $0 \le t' \le t < \infty$ as follows:

$$u_m(t) = \min_{0 \le t' \le t} u(t') \le u(t) \le 0 \qquad \text{(A3)}$$

It is obvious that $u_m(t)$ ($0 \le t$) is a monotone decreasing function.
Statement:

$$h(u_m(t)) = h\left(\min_{0 \le t' \le t} u(t')\right) = \min_{0 \le t' \le t} h(u(t')) \qquad \text{(A4)}$$

Proof: If $0 \le t' < t$ then the next inequalities stand:

$$u_m(t') \ge u_m(t) \Longrightarrow h(u_m(t')) \ge h(u_m(t))$$

since $u_m(t)$ is monotone decreasing and $h(x)$ is monotone increasing. The inequality on the left-hand side above stands for every $t' \in [0,t]$, consequently $h(u_m(t))$ equals the minimum of $h(u(t'))$, $t' \in [0,t]$. ☐

A2 Variance of the empirical reliability function

Statement
The variance of the empirical duplex normal reliability function, $r_{CT}(u)$ is given by:

$$\mathbb{D}^2(r_{CT}(u)) = \frac{1}{N} R_{CT}(u)[1 - R_{CT}(u)] \qquad \text{(A5)}$$

Proof
Consider the empirical normal reliability function, r_{CT}, given by Eq. (51) and rewrite it using Eq. (23):

$$r_{CT}(u) = \frac{1}{N}\sum_{n=1}^{N} \chi_{CT,n}(u) = \frac{1}{N}\sum_{n=1}^{N} \chi_{C,n}(u)\chi_{T,n}(u)$$

Considering that χ_C and χ_T are independent stochastic processes, the expectation of r_{CT} can be obtained as

$$\mathbb{E}(r_{CT}(u)) = \frac{1}{N}\sum_{n=1}^{N} \mathbb{E}\left(\chi_{C,n}(u)\chi_{T,n}(u)\right) = \mathbb{E}\left(\chi_{C,n}(u)\right)\mathbb{E}\left(\chi_{T,n}(u)\right) = R_C(u)\,R_T(u) \qquad \text{(A6)}$$

Hence, determining the variance of r_{CT}

$$\mathbb{D}^2(r_{CT}(u)) = \mathbb{E}\left(r_{CT}^2(u)\right) - \mathbb{E}^2(r_{CT}(u)) \qquad \text{(A7)}$$

needs only the calculation of the mean squared value, which is

$$\mathbb{E}\left(r_{CT}^2(u)\right) = \mathbb{E}\left(\frac{1}{N^2}\sum_{i,j}\chi_{CT,i}\chi_{CT,j}\right) = \mathbb{E}\left(\frac{1}{N^2}\sum_{i=j}\chi_{CT,i}\chi_{CT,j} + \frac{1}{N^2}\sum_{i\ne j}\chi_{CT,i}\chi_{CT,j}\right) = \frac{N}{N^2}\mathbb{E}(\chi_{CT,i}^2) + \frac{N^2-N}{N^2}\mathbb{E}(\chi_{CT,i})\mathbb{E}(\chi_{CT,j}) = \frac{1}{N}\mathbb{E}(\chi_{CT,i}) + \frac{N-1}{N}\mathbb{E}^2(\chi_{CT,i}) = \frac{1}{N}R_C(u)\,R_T(u) + \left(1 - \frac{1}{N}\right)R_C^2(u)\,R_T^2(u) \qquad \text{(A8)}$$

Substituting Eq. (A8) into Eq. (A7) and using Eq. (A6) provide the statement:

$$\mathbb{D}^2(r_{CT}(u)) = \frac{1}{N}R_C(u)R_T(u) + \left(1 - \frac{1}{N}\right)R_C^2(u)R_T^2(u) - R_C^2(u)R_T^2(u) =$$
$$= \frac{1}{N}R_C(u)R_T(u)[1 - R_C(u)R_T(u)] = \frac{1}{N}R_{CT}(u)[1 - R_{CT}(u)] \qquad ☐$$